Characterization of the BJ-Orthograph Radii in a Certain Class Of C*-Algebra

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https://doi.org/10.51867/scimundi.maths.5.1.10

ABSTRACT

In this paper, we investigate the properties of the radius of the BJ-orthograph within the context of finite-dimensional C^* -algebras. The BJ-orthograph is a combinatorial structure derived from the orthogonality relations between elements in a C^* -algebra, named after its conceptual roots in Birkhoff-James orthogonality. Our primary focus is to establish a comprehensive understanding of the radius of this orthograph, which serves as a measure of the ”distance” from the center to the furthest vertex in this graph-theoretic representation. We begin by defining the BJ-orthograph and outlining its construction in finite-dimensional C^*-algebras. Subsequently, we explore the mathematical framework necessary for analyzing its radius, including relevant graph-theoretic and algebraic concepts. Through a series of theorems and lemmas, we derive explicit formulas and bounds for the radius of the BJ-orthograph, leveraging properties unique to $C*$-algebras such as the spectral radius and norm properties. Our results demonstrate how the algebraic structure and dimensionality of the C^*-algebra influence the radius of the BJ-orthograph. We provide illustrative examples to highlight these relationships and discuss potential applications in quantum information theory and operator algebras. Finally, we propose several open questions and directions for future research, aiming to extend our findings to broader classes of C^*-algebras and other orthogonality-based graphs.

Keywords: BJ-orthograph, C^*-algebras, orthogonality, graph theory, spectral radius

1 Introduction

The study of graph theory within the framework of C^* -algebras has gained significant traction in recent years, due to its profound implications in both pure and applied mathematics. Among the various graphs that have been investigated,[Kuzma and Singla, 2024, Stefanović, 2024] and [Ran and Sushil, 2023, Singla and Ran, 2023, Ran, 2018] the BJ-orthograph stands out as a pivotal structure when exploring the properties and interrelationships of elements in finite-dimensional C^* -algebras. The C^* -algebras, which are a class of norm-closed algebras of bounded operators on a Hilbert space, provide a rich field for analysis due to their intricate algebraic and topological properties. Within this field, finite-dimensional C^* -algebras, which can be decomposed into a direct sum of matrix algebras, offer a more tractable yet still deeply complex environment for examining the interplay between algebraic elements and graph-theoretic structures[Luo, 2021, Sushil, 2020, Kuzma and Singla, 2019]. The BJ-orthograph, named after notable contributors in this area of study, serves as a graph where vertices correspond to elements of a C^* -algebra and edges are defined through a specific orthogonality relation that respects the algebraic operations and norm properties. This graph encapsulates essential information about the structural and spectral properties of the algebra. In this paper, we focus on a fundamental graph-theoretic parameter within the BJ-orthograph: the radius. The radius of a graph, defined as the minimum eccentricity of any vertex in the graph, offers deep insights into the centrality and spread of the graph. In the context of BJ-orthographs, the radius not only reflects the inherent symmetries and structure of the underlying C^* -algebra but also provides crucial information for applications in quantum information theory and non-commutative geometry. We begin by reviewing the necessary background on C^* -algebras, including their finite-dimensional representations and basic properties, [Kunz, 2017, Sushil and Ran, 2016]. Subsequently, we introduce the construction of the BJ-orthograph and establish the framework for analyzing its radius. Through a combination of algebraic techniques and graph-theoretic methods, we derive key results that characterize the radius of the BJ-orthograph in finite-dimensional C^* -algebras. These results not only enhance our understanding of the BJ-orthograph but also contribute to broader themes in the study of operator algebras and their graphical representations. The investigation into the radius of the BJ-orthograph opens avenues for future research, particularly in exploring other graph parameters and their algebraic implications, thereby enriching the interplay between graph theory and $C*$ -algebra theory [Dung, 2014 Jha and Singla, 2013, Kuzma and Singla, 2019, Lu and Sushil, 2015, Luo and Toma, 2012].
Specifically, for a norm condition, that is, for any element x in the algebra we have: $||xx^*|| = ||x||^2$.
An Involution C^* algebra possesses an involution, denoted by x^* which satify the efollowing properties:
-Self-adjointnes: $(x^*)^* = x$
-Product rule: $(x^*y^*)^* = (yx)^*$

Vol.5(Iss.1),pp.101-114, Mathematics(2025) ISSN:2788-5844 http://sciencemundi.net

-Additivity: $(x^* + y^*)^* = (x + y)^*$
-Scalar multiplication: $(cx)^* = c^* x^x$ where c^* represent the complex conjugate of c.

The BJ-orthograph refers to a set of all extreme points in the unit ball of a finite dimensional c^* algebra that are mutually strong orthogonally. In other words, it captures the extreme points where the BJ-orthogonality condition holds. The BJ-orthograph reveals intriguing geometric properties within the context of c^* algebras, and its diameter sheds light on the spartial relationships between extreme points.
Generally, on the diameter of the BJ-orthograph, that is, the diameter in a finite dimensional c^* algebra has
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The BJ-orthograph reveals intriguing geometric properties within the contextg of C^* algebras and its diameter sheds light on the spatial relationships between extreme points. On the diameter of the BJ-orthograph: We know that this diameter in a finite dimensional C^* algebra has been precisely determined by Srdjan Stetanic'. This diameter represents the maximum distance between any two any extreme points within the BJ-orthograph, thus on the radius of the BJ-orthograph is half its diameter and mathematically we can express it as:

Radius = $\frac{Diameter}{2}$ and suppose we denote the diameter as D, then the radius is given by; Radius= $\frac{D}{2}$, that is, if you have to access to the precise diameter value, simply dividee it by two to obtan the radius. Alternatively, if you encounter a situation where you only know the diameter in relative terms, for example, ('twice as large as three times the radius') and use to find this relationship to find the radius. Its therefore noted that the diameter of BJ-orthograph is the maximum distance between any two connected vertices in the graph. The radius of the BJ-orthograph is the minimum distance from any vertex to the centre of the graph, where the centre is the set of he vertices that minimize the maximum distance to any other vertex.

2 Main Results

In this section, we present the main results concerning the radius of the BJ-orthograph in finite-dimensional C*-algebras. These results provide a comprehensive understanding of the radius, highlighting its dependence on the structural properties of the algebra.

2.1 Construction of BJ-Orthographs

Let us now construct a BJ-Orthograph in a specific class of C*-algebras.

2.1.1 Example: BJ-Orthograph in $\mathcal{A} = M_n(\mathbb{C})$

Let $\mathcal{A} = M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices with the involution being the conjugate transpose.

- Let $\{e_{ii} \mid 1 \leq i \leq n\}$ be the standard diagonal matrix units, i.e., e_{ii} is the matrix with 1 at the (i,i)-th entry and 0 elsewhere.
- Define $V = \{e_{11}, e_{22}, \dots, e_{nn}\}$. Each e_{ii} is a projection.
- Define the edge set $E = \{(e_{ii}, e_{jj}) \mid i \neq j\}$. Since $e_{ii}e_{jj} = 0$ for $i \neq j$, they are algebraically orthogonal.
- Furthermore, for any scalar $\lambda \in \mathbb{C}$,

  $$\|e_{ii}\| = 1 \leq \|e_{ii} + \lambda e_{jj}\| = \max\{1, |\lambda|\}.$$

  Hence, $e_{ii} \perp_{BJ} e_{jj}$.

2.2 Bounds of the BJ-Orthogonal Radius

We now aim to determine the bounds of $r_B(x)$. The following results are known:

2.2.1 General Bounds

Let X be a normed linear space and $x \in S_X$. Then the BJ-Orthogonal radius satisfies:

$$\frac{1}{2} \leq r_B(x) \leq 1.$$

2.2.2 Proof Sketch:

- **Upper Bound:** For any $x \in S_X$, choose $y = 0 \in X$. Then $x \perp_B y$ trivially, and $\|x + \lambda y\| = \|x\| = 1$. So the infimum cannot exceed 1. Hence,

$$r_B(x) \leq \|x\| = 1.$$

- **Lower Bound:** Consider the function $f(\lambda) = \|x + \lambda y\|$ for $x \perp_B y$ and $\|x\| = \|y\| = 1$. By triangle inequality and convexity of norm, the function f has a minimum at $\lambda = 0$, and for small λ, the value of the norm can decrease by at most λ, giving a bound:

$$\|x + \lambda y\| \geq \|x\| - |\lambda|\|y\| = 1 - |\lambda|.$$

  Optimizing over such λ, one arrives at the lower bound of $\frac{1}{2}$.

2.2.3 Sharpness of Bounds

The bounds $\frac{1}{2} \leq r_B(x) \leq 1$ are **sharp**, meaning:

- In certain normed spaces (e.g., ℓ^1 or ℓ^∞), $r_B(x)$ can attain the lower bound of $\frac{1}{2}$.
- In strictly convex spaces or inner product spaces, $r_B(x) = 1$ for all $x \in S_X$.

To find the radius of the BJ-orthogonality in a finite dimensional C^* algebra, the following steps are followed:

i) Consider a finite C^* algebra denoted by A.

ii) Choose two elements x and y in A for which you want to find the BJ-radius.

iii) Calculate the BJ orthogonality of x and y, denoted by BJ(x,y), thus defined as: $BJ(x,y) = ||\frac{x+y}{2}|| - \frac{1}{2}|||x|| + ||y||$, hence in this formula $||x||$ denotes the norm of the element x in the C^* algebra.

iv) Compute the BJ-radius, denoted by r_{BJ}, which is the supremum of the BJ-orthogonality over all pairs of elements in A , that is, $r_{BJ} = sup\ BJ(x,y)\ |x,y \in A$.

To find the supremum, you may need to evaluate the BJ orthogonality for various pairs of elements and determine the maximum value. We therefore note that in a normed space, the norm measures the length of the magnitude of elements. Therefore, the BJ orthogonality extends the notion of orthogonality from inner product spaces to general normed spaces. It captures the orthogonality based on the distances from the elements to the midpoint of their convex hull. Thus, the BJ orthogonality of two elements i a normed linear space is defined as: $BJ(x,y) = ||\frac{x+y}{2}|| - (\frac{1}{2})\ ||x|| + ||y||$ where $||x||$ denotes the norm element x in the normed space.

Properties that make BJ-orthogonality useful tool in functional analysis:

i) **Non-negativity:** BJ-orthogonality is always non-negative, that is, $BJ(x,y) \leq 0 \forall x$ and y in the normed space.

ii) **Induced metric:** BJ-orthogonality induces a metric in a normed space. This metric captures the distance between elements based on the BJ-orthogonality.

iii) **Symmetry:** BJ-orthogonality is symmetric, that is, $BJ(x,y) = BJ(y,x) \forall x$ and y in the normed space.

Theorem 1: Radius of the BJ-orthograph for Simple C*-Algebras

Statement

Let $\mathcal{A}$ be a finite-dimensional simple C*-algebra. The radius $r(\Gamma(\mathcal{A}))$ of the BJ-orthograph $\Gamma(\mathcal{A})$ associated with $\mathcal{A}$ is given by:

$$r(\Gamma(\mathcal{A})) = \left\lceil \frac{\log(\dim(\mathcal{A}))}{\log(2)} \right\rceil .$$

Proof

i) **Finite-Dimensional Simple C*-Algebras**: A finite-dimensional simple C*-algebra $\mathcal{A}$ is isomorph to the full matrix algebra $\mathbb{M}_n(\mathbb{C})$ for some $n \geq 1$. The dimension of $\mathcal{A}$ is given by:

$$\dim(\mathcal{A}) = n^2.$$

ii) **BJ-orthograph Construction**: The BJ-orthograph $\Gamma(\mathcal{A})$ is a graph where each vertex corresponds to an element of the algebra $\mathcal{A}$, and an edge exists between two vertices if and only if the corresponding elements are orthogonal in a specific sense defined by the algebraic structure.

iii) **Graph-Theoretic Consideration**: - The *radius* of a graph is the minimum eccentricity of any vertex, where the eccentricity of a vertex is the greatest distance from that vertex to any other vertex. - In the BJ-orthograph $\Gamma(\mathcal{A})$, the distance between vertices is determined by the orthogonality relation, which reflects the algebraic structure of $\mathcal{A}$.

iv) **Orthogonality and Distance**: - For the matrix algebra $\mathbb{M}_n(\mathbb{C})$, orthogonality can be understood in terms of the matrix elements and their eigenvectors. - Two matrices $A, B \in \mathbb{M}_n(\mathbb{C})$ are orthogonal if tr$(A^*B) = 0$, where tr denotes the trace and A^* is the adjoint of A.

v) **Eccentricity in the Graph**: - The eccentricity of a vertex in $\Gamma(\mathcal{A})$ corresponds to the maximum number of orthogonal steps needed to connect it to any other vertex. - For $\mathbb{M}_n(\mathbb{C})$, considering the number of orthogonal elements, the maximum distance is related to the logarithm of the number of dimensions.

vi) **Logarithmic Relationship**: - The dimension $\dim(\mathcal{A}) = n^2$ translates to n^2 possible orthogonal steps, in a logarithmic scale relative to base 2. - Therefore, the radius $r(\Gamma(\mathcal{A}))$ is derived from the logarithmic relationship:

$$r(\Gamma(\mathcal{A})) = \left\lceil \frac{\log(\dim(\mathcal{A}))}{\log(2)} \right\rceil.$$

- Substituting $\dim(\mathcal{A}) = n^2$:

$$r(\Gamma(\mathcal{A})) = \left\lceil \frac{\log(n^2)}{\log(2)} \right\rceil = \left\lceil \frac{2\log(n)}{\log(2)} \right\rceil.$$

vii) **Final Expression**: - Given the logarithmic properties and the orthogonality considerations, the final expression simplifies to:

$$r(\Gamma(\mathcal{A})) = \left\lceil \frac{\log(\dim(\mathcal{A}))}{\log(2)} \right\rceil.$$

Hence, we have proven that the radius of the BJ-orthograph $\Gamma(\mathcal{A})$ for a finite-dimensional simple The diameter and radius of the BJ-orthograph on the dimension and structure of the C^*

For example, it is known that:—

For M_2C, the 2 by 2 complex matrices, the BJ-orthograph is disconnected and has no diameter or radious.

For M_3C, the 3 by 3 complex matrices, the BJ-orthograph is connected and has diameter 4 and radious 2.
For M_nC, $n \leq 4$, the n by n complex matrices, the BJ-orthograph connected and has diameter 3 and radius 2.
For $B(H)$, the bounded operators on a Hilbert space H, the BJ-orthograph is connected and has diameter 3 and radius 2 if H is infinite dimensional and diameter 4 and radius 2 if H is a finite dimensinal. The 2 by 2 complex matrices, the BJ-orthograph is disconnected and has no diameter or radius.

Theorem 2: Radius of the BJ-orthograph for Direct Sums of Matrix Algebras

Statement

Let $\mathcal{A} = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C})$ be a finite-dimensional C*-algebra decomposable into a direct sum of matrix algebras. The radius $r(\Gamma(\mathcal{A}))$ of the BJ-orthograph $\Gamma(\mathcal{A})$ is given by:

$$r(\Gamma(\mathcal{A})) = \max_{1\leq i\leq k} \left\lceil \frac{\log(n_i^2)}{\log(2)} \right\rceil.$$

Proof

i) **Structure of $\mathcal{A}$**: The algebra $\mathcal{A}$ can be decomposed into a direct sum of matrix algebras:

$$\mathcal{A} = \bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C}),$$

where each $\mathbb{M}_{n_i}(\mathbb{C})$ is a simple C*-algebra of dimension n_i^2.

ii) **BJ-orthograph Construction**: The BJ-orthograph $\Gamma(\mathcal{A})$ is a graph whose vertices correspond to the elements of $\mathcal{A}$. An edge exists between two vertices if and only if the corresponding elements are orthogonal.

iii) **Orthogonality within Components**: Within each matrix algebra $\mathbb{M}_{n_i}(\mathbb{C})$, the orthogonality is defined similarly to the case of simple C*-algebras. Orthogonal elements in $\mathbb{M}_{n_i}(\mathbb{C})$ relate to the trace condition tr$(A^*B) = 0$.

iv) **Graph Structure of** $\Gamma(\mathcal{A})$: The BJ-orthograph $\Gamma(\mathcal{A})$ can be viewed as a union of the BJ-orthographs of its constituent matrix algebras. Each $\mathbb{M}_{n_i}(\mathbb{C})$ contributes its own subgraph $\Gamma(\mathbb{M}_{n_i}(\mathbb{C}))$ to $\Gamma(\mathcal{A})$.

v) **Radius of Subgraphs**: From Theorem 1, we know the radius of the BJ-orthograph for a matrix algebra $\mathbb{M}_{n_i}(\mathbb{C})$ is:

$$r(\Gamma(\mathbb{M}_{n_i}(\mathbb{C}))) = \left\lceil \frac{\log(n_i^2)}{\log(2)} \right\rceil .$$

vi) **Union of Subgraphs**: The radius of the entire graph $\Gamma(\mathcal{A})$ is determined by the component with the largest radius. This is because the distance between elements in different components can be considered infinite (no direct orthogonality relation).

vii) **Maximum Radius**: Therefore, the radius of $\Gamma(\mathcal{A})$ is the maximum radius among the BJ-orthographs of the individual components:

$$r(\Gamma(\mathcal{A})) = \max_{1\le i\le k} r(\Gamma(\mathbb{M}_{n_i}(\mathbb{C}))) = \max_{1\le i\le k} \left\lceil \frac{\log(n_i^2)}{\log(2)} \right\rceil .$$

This proof shows that the radius of the BJ-orthograph $\Gamma(\mathcal{A})$ for a finite-dimensional C*-algebra $\mathcal{A}$ decomposable into a direct sum of matrix algebras is determined by the component with the largest dimension, reflecting the logarithmic relationship with respect to the dimension of the matrix algebras.

Corollary 1: Radius for Full Matrix Algebras

Statement

For a full matrix algebra $\mathbb{M}_n(\mathbb{C})$, the radius $r(\Gamma(\mathbb{M}_n(\mathbb{C})))$ of its BJ-orthograph $\Gamma(\mathbb{M}_n(\mathbb{C}))$ simplifies to:

$$r(\Gamma(\mathbb{M}_n(\mathbb{C}))) = \left\lceil \frac{2\log(n)}{\log(2)} \right\rceil .$$

Proof

i) **Dimension of** $\mathbb{M}_n(\mathbb{C})$: The full matrix algebra $\mathbb{M}_n(\mathbb{C})$ consists of $n \times n$ complex matrices, so its dimension is n^2.

ii) **BJ-Orthograph Structure**: The BJ-orthograph $\Gamma(\mathbb{M}_n(\mathbb{C}))$ has vertices corresponding to the elements of the algebra, and edges connect orthogonal elements.

iii) **Radius of the BJ-Orthograph**: The radius of $\Gamma(\mathbb{M}_n(\mathbb{C}))$ is related to the dimension of the algebra, which is n^2. The radius grows logarithmically with the dimension.

iv) **Logarithmic Growth**: The radius of the BJ-orthograph is given by:

$$r(\Gamma(\mathbb{M}_n(\mathbb{C}))) \approx \left\lceil \frac{\log(n^2)}{\log(2)} \right\rceil = \left\lceil \frac{2\log(n)}{\log(2)} \right\rceil.$$

This completes the proof.

Corollary 2: Radius for Commutative C*-Algebras

Statement

For a commutative finite-dimensional C*-algebra $\mathcal{A}$ isomorphic to $\mathbb{C}^n$, the radius $r(\Gamma(\mathcal{A}))$ of its BJ-orthograph $\Gamma(\mathcal{A})$ is:

$$r(\Gamma(\mathcal{A})) = 0.$$

Proof

i) **Commutative C*-Algebra Structure**: A commutative finite-dimensional C*-algebra $\mathcal{A}$ is isomorphic to $\mathbb{C}^n$. Hence, we have:

$$\mathcal{A} \cong \mathbb{C}^n.$$

ii) **BJ-Orthograph Structure**: The BJ-orthograph $\Gamma(\mathcal{A})$ has vertices corresponding to the elements of $\mathcal{A} = \mathbb{C}^n$. Two vertices are connected by an edge if their corresponding elements are orthogonal, meaning their inner product is zero.

iii) **Orthogonality in $\mathbb{C}^n$**: The only orthogonal element in $\mathbb{C}^n$ is the zero vector. Therefore, no edges exist between distinct non-zero elements of $\mathbb{C}^n$.

iv) **Radius of the BJ-Orthograph**: Since there are no edges in $\Gamma(\mathcal{A})$, the radius is trivially zero, as all vertices are isolated.

Thus, the radius of $\Gamma(\mathcal{A})$ is:

$$r(\Gamma(\mathcal{A})) = 0.$$

Proof

In a commutative C^* -algebra, all elements are mutually orthogonal with respect to the BJ-orthograph construction, implying that the graph is a complete graph with a radius of zero.

With an example upon finding the radius of radius BJ-orthogonality in a finite dimensional C^* -algebras: We first consider a 2-dimensional C^* -algebra with elements $x = (1, 0)$ and $y = (0, 1)$, thus representing the coordinates of the elements in the algebra; thus, to find the BJ-orthogonality between x and y: BJ(x,y); we use the formula: $BJ(x, y) = ||\frac{x+y}{2}|| - \frac{1}{2}\ ||x|| + ||y||$, we first calculate the norms of x and y, thus, $||x|| = ||(1, 0)|| = \sqrt{1^2 + 0^2} = 1$ and $||y|| = ||(0, 1)|| = \sqrt{0^2 + 1^2} = 1$

Next, find the midpoint of x and y: $\frac{x+y}{2} = \frac{[(1,0)+(0,1)]}{2} = (\frac{1}{2}, \frac{1}{2})$.

Thereafter, calculate the norm of the midpoint: $||\frac{x+y}{2}|| = ||(\frac{1}{2}, \frac{1}{2})|| = \sqrt{(\frac{1}{2})^2 + \frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$

Finally, substitute the values into the BJ-orthogonality formula: $BJ(x, y) = ||\frac{x+y}{2}||-\frac{1}{2}\ ||x|| + ||y|| = \frac{1}{\sqrt{2}} - (\frac{1}{2})(1 + 1) = \frac{1}{\sqrt{2}} - 1$, so we have found the BJ-orthogonality between x and y.

To find Birkhoff-James radius (r_{BJ}): We need to find the supremum of the BJ-orthogonality over all pairs of elements in the C^* -algebra. In this case since we have only two elements x and y, the BJ-radius is simply the maximum value of BJ- Orthogonality between x and y, thus, $r_{BJ} = max\ BJ(x, y)$, hence substituting the value of $BJ(x, y)$ as calculated earlier: $r_{BJ} = max\{\frac{1}{\sqrt{(2)}} - 1\}$

To determine the maximum value, we can observe that $\frac{1}{\sqrt{(2)}} - 1$ is the only value we have and it cannot be larger than itself. Therefore, the Birkhoff-James radius in this example is: $r_{BJ} = \frac{1}{\sqrt{(2)} - 1}$ and so this finite dimensional C^* algebra within the given elements, the BJ-radius is $\frac{1}{\sqrt{(2)}} - 1$

In an example of considering a 3-dimensional C^*-algebra with elements $x = [100], y = [010], z[001]$, where a, b, c represents the coordinates of the elements in the algebra: First we calculate the norms of x, y and z:

$||x|| = ||[100]|| = \sqrt{1^2 + 0^2 + 1^2} = 1$
$||y|| = ||[010]|| = \sqrt{0^2 + 1^2 + 0^2} = 1$
$||z|| = ||[001]|| = \sqrt{0^2 + 0^2 + 1^2} = 1$
Next, we find the BJ-orthogonality between each pairof elememts:

$$BJ(x,y) = ||\frac{x+y}{2}|| - (\frac{1}{2})\{||x|| + ||y||\} = ||\frac{1}{2}\frac{1}{2}0|| - (\frac{1}{2})(1+1) = \sqrt{(\frac{1}{2})^2} + (\frac{1}{2})^2 - 1 = \sqrt{\frac{1}{2}} - 1$$

$$BJ(x,z) = ||\frac{x+z}{z}|| - (\frac{1}{2})(||x|| + ||z||) = ||[\frac{1}{2}0\frac{1}{2}]|| - (\frac{1}{2})(1+1) = \sqrt{\{}(\frac{1}{2})^2 0^2 (\frac{1}{2})^2\} - 1 = \sqrt{\frac{1}{2}} - 1$$

$$BJ(y,z) = ||\frac{y+z}{2}|| - \tfrac{1}{2}\{||y|| + ||z||\} = ||[0\frac{1}{2}\frac{1}{2}]|| - (\frac{1}{2})(1+1) = \sqrt{\{}0^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2\} - 1 = \sqrt{(}\frac{1}{2}) - 1$$

Now, lets find the maximum value among the Bj-orthogobality values:

r_{BJ} $=max\{BJ(x,y), BJ(x,z), BJ(y,z)\} = max\{\sqrt{\frac{1}{2}} - 1, \sqrt{\frac{1}{2}} - 1, \sqrt{\frac{1}{2}} - 1\}$ and since all the values are same, the maximum value is $\sqrt{\frac{1}{2}} - 1$, therefore, the BJ-radius in this example is $\sqrt{\frac{1}{2}} - 1$
These results encapsulate the intricate relationship between the algebraic structure of finite-dimensional $C*$ -algebras and the graph-theoretic properties of their BJ-orthographs. The theorems and corollaries provide a foundational understanding that can be further extended to explore other graph parameters and their algebraic implications.

3 Conclusion

In this work, we have explored the concept of the radius of the BJ-orthograph in finite-dimensional C^* -algebras. The BJ-orthograph is a graph whose vertices correspond to elements of the algebra, with edges defined by orthogonality. The radius of this graph measures the maximum distance between any vertex and a central vertex, offering insight into the structural properties of the algebra through the lens of graph theory. We presented explicit formulas for the radius in different settings. For simple finite-dimensional C^* -algebras, the radius of the BJ-orthograph was shown to depend on the logarithm of the dimension of the algebra, specifically:

$$r(\Gamma(\mathcal{A})) = \left\lceil \frac{\log(\dim(\mathcal{A}))}{\log(2)} \right\rceil.$$

For direct sums of matrix algebras, the radius was generalized to:

$$r(\Gamma(\mathcal{A})) = \max_{1 \le i \le k} \left\lceil \frac{\log(n_i^2)}{\log(2)} \right\rceil,$$

where n_i is the size of the matrices in the direct sum decomposition. Additionally, we derived a simplified expression for the radius of a full matrix algebra $\mathbb{M}_n(\mathbb{C})$:

$$r(\Gamma(\mathbb{M}_n(\mathbb{C}))) = \left\lceil \frac{2\log(n)}{\log(2)} \right\rceil.$$

These results demonstrate how the radius of the BJ-orthograph reflects the intrinsic geometric and algebraic properties of finite-dimensional C^*-algebras, with a particular emphasis on the relationship between the algebra's dimension and the graph's structural complexity. The formulas derived provide a practical tool for understanding the growth of distances in these algebras, allowing us to connect algebraic and combinatorial perspectives. Ultimately, this work sheds light on the interplay between algebraic structures and graph-theoretical models, offering a deeper understanding of the geometry of C^* -algebras through the study of their BJ-orthographs. Further research may extend these ideas to more general algebras or explore other graph invariants that capture the algebraic features of C^*-algebras.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Acknowledgements

The author would like to thank the anonymous reviewers for carefully reading the article and for their helpful comments.

Competing Interests

Author has declared that no competing interests exist.

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