Vol 3(Iss.1),pp.135-142,2023 Science Mundi ISSN:2788-5844 http://sciencemundi.net

On Generalized Sums of Six, Seven and Nine Cube

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ABSTRACT

Let $u_1, u_2, u_3, \cdots u_n$ be integers such that $u_n - u_{n-1} = u_{n-1} - u_{n-2} = \cdots = a_2 - a_1 = d$. In this article, the study of sums of cube in arithmetic progression is discussed. In particular, the study develops and introduces some generalized results on sums of six, seven and nine cube for any arbitrary integers in arithmetic sequences.The method of study involves analogy grounded on integer decomposition and factorization. The result in this study will prove the existing results on sums of cubes.

Keywords: Diophantine Equation; Sums of Six, Seven and Nine Cube

1 Introduction

The study of integer decomposition into sums of cubes is a historical problem and has been a subject of mathematical inquiry for a long time now. This fascinating intellectual undertaking was pioneered in 18th century by Edward Waring, in his seminal work ”Meditationes Algebraicae” (1770). Edward started the marvellous work of such representations setting the ground for future explorations. Great scholars in mathematics like Euler [4] and Lagrange [11] made notable strides in conceptualizing this long standing problem. Lagrange’s method for determining the representations of integers as sums of four squares is a big step to his contributions to Waring’s problem [11]. In the 20th century, great mathematicians like David Hilbert [6], with his well known famous list of 23 problems, attracted attention to Waring’s problem by generalizing some important results. Historical mathematicians like Ramanujan [13] and Davenport [3], contributed very significant results that led to enrich the theory of partitions at large within number theory, contributing to the results on this mathematical quest.

For Some contributions on study of diophantine equation of degree less than 5, the reader can consult [2, 5, 12, 14] and for detailed recap on polynomial equations on sums of squares the reader may survey [7, 8, 9, 10]. Recently, the scholar Booker [1] showed that computational techniques have led to the discovery of some new solutions on sums of cubes. This historical contributions, with each mathematician's unique contributions, illustrates much devoted journey on the problem of integer representation as sums of cubes. Recent research in the field of integer decomposition as sums of cubes in arithmetic sequences have been done in [9]. In this study, a detailed generalization of integer representations as sums of three, four, and five cubes were completely characterized. In the current study, the research aims at expanding on this integer classification of sums of cube, by focusing on integer decompositions as sums of six, seven, and nine cubes in arithmetic progressions, while considering non-zero integer solutions. The study will contribute to deep understanding of integer solutions in cubic sums, providing insights into the intricate connections between the cubes and their alignment in arithmetic progression.

2 Main Results

In this section we provide our findings and explore specific instances of our formula, denoted by

$$I = (u_1 + u_2 + u_3 + \cdots + u_n)L = u_1^3 + u_2^3 + u_3^3 + \cdots + u_n^3,$$

such that $u_n > u_{n-1} > \cdots > u_1$.
This is done on a case by case basis as follows:

Case $I : n = 6$

Proposition 2.1. $I = (u_3^2 + 11d^2 + u_1 d)\sum_{i=1}^{6} u_i = \sum_{i=1}^{6} u_i^3$ *has solution in integers if* $u_{i+1} - u_i = d$ *for* $1 \leq i \leq 5$*.*

Proof. Let $u_{i+1} - u_i = d$ for $1 \leq i \leq 5$. This implies, $u_2 = u_1 + d, u_3 = u_1 + 2d, u_4 = u_1 + 3d, u_5 = u_1 + 4d, u_6 = u_1 + 5d$.
To prove

$$(u_3^2 + 11d^2 + u_1 d)\sum_{i=1}^{6} u_i = \sum_{i=1}^{6} u_i^3 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2.1)$$

We can proceed from right hand side of equation 2.1.
Now,

$$(u_3^2 + 11d^2 + u_1 d)(u_1 + u_2 + u_3 + u_4 + u_5 + u_6)$$
$$= ((u_1 + 2d)^2 + 11d^2 + u_1 d)(u_1 + (u_1 + d)(u_2 + 2d)(u_3 + 3d)(u_4 + 4d)(u_5 + 5d))$$

$$= (u_1^2 + 4u_1d + 4d^2 + 11d^2 + u_1d)(6u_1 + 15d)$$
$$= 6u_1(u_1^2 + 4u_1d + 4d^2 + 11d^2 + u_1d)(6u_1 + 15d)$$
$$6u_1(u_1^2 + 4u_1d + 4d^2 + 11d^2 + u_1d) + 15d(u_1^2 + 4u_1d + 4d^2 + 11d^2 + u_1d)$$
$$u_1^3 + 30u_1^2d + 90u_1d^2 + 15u_1^2d + 75u_1d^2 + 225d^3$$
$$= 6u_1^3 + 45u_1^2d + 165u_1d^2 + 225d^3 \cdots\cdots\cdots\cdots\cdots\cdots (2.2)$$

Decomposing equation (2.2) into sums of cube we have:

$$6u_1^3 + 45u_1^2d + 165u_1d^2 + 225d^3$$
$$= u_1^3 + (u_1 + d)^3 + (u_1 + 2d)^3 + (u_1 + 3d)^3 + (u_1 + 4d)^3 + (u_1 + 5d)^3$$
$$= u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 + u_6^3$$
$$= \sum_{i=1}^{6} u_i^3$$

as desired.

□

Case $II : n = 7$

Proposition 2.2. *$I = (u_1^2 + 6u_1d + 21d^2)\sum_{i=1}^{7} u_i = \sum_{i=1}^{7} u_i^3$ has solution in integers if $u_{i+1} - u_i = d$ for $1 \leq i \leq 6$*

Proof. Let $u_{i+1} - u_i = d$ for $1 \leq i \leq 6$. Then $u_2 = u_1 + d, u_3 = u_1 + 2d, u_4 = u_1 + 3d, u_5 = u_1 + 4d, u_6 = u_1 + 5d, u_7 = u_1 + 6d$.
To prove

$$(u_1^2 + 6u_1d + 21d^2)\sum_{i=1}^{7} u_i = \sum_{i=1}^{7} u_i^3 \cdots\cdots\cdots\cdots\cdots\cdots (2.3)$$

We can proceed from right hand side of equation (2.3). Now,

$$(u^2 + 6u_1d + 21d^2)(u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7)$$
$$= ((u_1 + 2d^2) + 6u_1d + 21d^2)(u_1 + (u_1 + d) + (u_2 + 2d) + (u_3 + 3d) + (u_4 + 4d) + (u_5 + 5d) + (u_6 + 6d))$$
$$= (u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7)(u^2 + 6u_1d + 21d^2)$$
$$= (u^2 + 6u_1d + 21d^2)(7u_1 + 21d)$$
$$= 7u_1(u^2 + 6u_1d + 21d^2) + 21d(u^2 + 6u_1d + 21d^2)$$

$$= 7u_1^3 + 42u_1^d d + 147u_1d^2 + 21u_1^2d + 126u_1d^2 + 441d^3$$
$$= 7u_1^3 + 63u_1^2d + 273u_1d^2 + 441d^3 \cdots\cdots\cdots\cdots\cdots (2.4)$$

Decomposing equation (2.4) into sums of cube we have:

$$7u_1^3 + 63u_1^2d + 273u_1d^2 + 441d^3$$
$$= u_1^3 + (u_1+d)^3 + (u_1+2d)^3 + (u_1+3d)^3 + (u_1+4d)^3 + (u_1+5d)^3 + (u_1+6d)^3$$
$$= u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 + u_6^3 + u_7^3$$
$$= \sum_{i=1}^{7} u_i^3.$$

This complete the proof.

□

Proposition 2.3. $I = (u_2^2 + 6u_2d + 29d^2 + 22d^2)\sum_{i=1}^9 u_i = \sum_{i=1}^9 u_i^3$ *has solution in integers if* $u_{i+1} - u_i = d$ *for* $1 \le i \le 8.$

Proof. Let $u_{i+1} - u_i = d$ for $1 \le i \le 8$. Then $u_3 = u_2 + d, u_4 = u_2 + 2d, u_5 = u_2 + 3d, u_6 = u_2 + 4d, u_7 = u_2 + 5d, u_8 = u_2 + 6d, u_9 = u_2 + 7d$. To prove that,

$$(u_2^2 + 6u_2d + 29d^2)\sum_{i=1}^{9} u_i = \sum_{i=1}^{9} u_i^3 \cdots\cdots\cdots\cdots\cdots (2.5)$$

We can proceed from right hand side of equation (2.5). Now $I = (u_2^2 + 6u_2d + 29d^2)(u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9)$= $(u_2^2 + 6u_2d + 29d^2)\sum_{i=1}^9 u_i$

$$= (u_2^2+6u_2d+29d^2)((u_2-d)+u_2+(u_2+d)+(u_2+2d)+(u_2+3d)+(u_2+4d)+(u_2+5d)+(u_2+6d)+(u_2+7d))$$
$$= (u_2^2 + 6u_2d + 29d^2)(9u_2 + 27d)$$
$$= 9u_2^3 + 81u_2^2d + 423u_2d^2 + 783d^3 \cdots\cdots\cdots\cdots\cdots\cdots (2.6)$$

Decomposing equation (2.6) into sums of cube we have:

$$9u_2^3 + 81u_2^2d + 423u_2d^2 + 783d^3$$
$$= ((u_2-d)^3 + u_2^3 + (u_2+d)^3 + (u_2+2d)^3 + (u_2+3d)^3 + (u_2+4d)^3 + (u_2+5d)^3 + (u_2+6d)^3 + (u_2+7d)^3)$$
$$= u_1^3 + u_2^3 + u_3^3 + u_4^3 + u_5^3 + u_6^3 + u_7^3 + u_8^3 + u_9^3 = \sum_{i=1}^{9} u_i^3$$

This establishes the proof.

□

In this subsection, we provide some examples to support our results in proposition 2.1, 2.2 and 2.3. We have the following :

Table 1: Sums of Six Cubes

u_1^3	u_2^3	u_3^3	u_4^3	u_5^3	u_6^3	$u_1^3+u_2^3+u_3^3+u_4^3+u_5^3+u_6^3 = I = (u_1+u_2+u_3+u_4+u_5+u_6)(u_3^2+11u_1d^2+u_1d)$	d
5^3	6^3	7^3	8^3	9^3	10^3	2925	1
11^3	13^3	15^3	17^3	19^3	21^3	27936	2
17^3	20^3	23^3	26^3	29^3	32^3	99813	3
23^3	27^3	31^3	35^3	39^3	43^3	243342	4
35^3	41^3	47^3	53^3	59^3	65^3	844500	6
60^3	70^3	80^3	90^3	100^3	110^3	4131000	10

Table 2: Sums of Seven Cubes

u_1^3	u_2^3	u_3^3	u_4^3	u_5^3	u_6^3	u_7^3	$u_1^3+u_2^3+u_3^3+u_4^3+u_5^3+u_6^3+u_7^3 = I = (u_1+u_2+u_3+u_4+u_5+u_6+u_7)(u_2^2+6u_1d+21d^2)$	d
5^3	6^3	7^3	8^3	9^3	10^3	11^3	4256	2
3^3	9^3	15^3	21^3	27^3	33^3	39^3	128331	6
5^3	10^3	15^3	20^3	25^3	30^3	35^3	98000	5
68^3	79^3	90^3	101^3	112^3	123^3	134^3	8238671	11
19^3	25^3	31^3	37^3	43^3	49^3	55^3	466459	6
29^3	40^3	51^3	62^3	73^3	84^3	95^3	2298464	11
12^3	19^3	26^3	33^3	40^3	47^3	54^3	387387	7
46^3	55^3	64^3	73^3	82^3	91^3	100^3	3219811	9
1^3	3^3	5^3	7^3	9^3	11^3	13^3	4753	2

Table 3: Sums of Nine Cubes

u_1^3	u_2^3	u_3^3	u_4^3	u_5^3	u_6^3	u_7^3	u_8^3	u_9^3	$u_1^3+u_2^3+u_3^3+u_4^3+u_5^3+u_6^3+u_7^3+u_8^3 = I = (u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8)(u_1^2d^2+27u_1^2+7u_1+1)$	d
17^3	18^3	19^3	20^3	21^3	22^3	23^3	24^3	25^3	71504	1
10^3	25^3	40^3	55^3	70^3	85^3	100^3	115^3	130^3	3725000	15
20^3	50^3	70^3	90^3	110^3	130^3	150^3	170^3	190^3	13040000	20
25^3	26^3	27^3	28^3	29^3	30^3	31^3	32^3	33^3	188784	1
2^3	5^3	8^3	11^3	14^3	17^3	20^3	23^3	26^3	29800	3
3^3	7^3	11^3	15^3	19^3	23^3	27^3	31^3	35^3	73576	4
40^3	47^3	54^3	61^3	68^3	75^3	82^3	89^3	96^3	2544912	7
4^3	10^3	16^3	22^3	28^3	34^3	40^3	46^3	52^3	238400	6
10^3	19^3	28^3	37^3	46^3	55^3	64^3	73^3	82^3	995336	9

3 Conclusion

In summary, the study of arithmetic progressions involving the sum of six, seven and nine has been investigated in this research. The patterns and relationships unveiled in this research and can have broader implications in number theory and more especially in implications of Diophantine equations in cryptography. To advance this research, one can investigate similar arithmetic progressions involving sums of cube for numbers beyond six, seven and nine. Exploring higher powers, such as fourth or fifth powers, could unveil more intricate patterns and relationships. One can also seek to develop general formulas or algorithms for calculating n^{th} term of such arithmetic progressions, making it easier to predict and compute values for various powers.

Competing Interests Authors have declared no competing interest.

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