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A Study of orthogonal projectors and their fundamental properties in Hilbert spaces

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Abstract

This paper presents a systematic study of orthogonal projectors in Hilbert spaces and examines their fundamental properties within the framework of functional analysis. Basic concepts related to projectors, orthogonal complements, convex subsets, and direct sums are reviewed and developed. The existence and uniqueness of best approximations from closed convex subsets of a Hilbert space are established using the parallelogram law. The projection theorem is then proved, showing that every element of a Hilbert space H can be uniquely decomposed into the sum of an element of a closed subspace M and an element of its orthogonal complement $M^{\perp}$. As a consequence, the identity $M^{\perp\perp} = M$ is established for every closed linear subspace M. Additional characterizations of dense subspaces through orthogonality conditions are presented. The results provide a theoretical foundation for applications in numerical analysis, linear algebra, approximation theory, Gram–Schmidt orthogonalization, QR decomposition, and orthogonal polynomial theory.

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1 Introduction

Projectors, particularly orthogonal projections in Hilbert spaces, play a fundamental role in both theoretical and applied mathematics. The uniqueness of the orthogonal projector onto a closed subspace, together with the celebrated projection theorem, forms a cornerstone of Hilbert space theory. These concepts have been studied extensively because of their wide range of applications, including least squares regression, signal processing, quantum mechanics, and numerical linear algebra.

Damle, Lin, and Ying [3] discussed the explicit construction of orthogonal projectors, showing that for a given unit vector u, the orthogonal projector onto span$\{u\}$ is uniquely determined. In regression analysis, the method of least squares is equivalent to projecting the observation vector of the dependent variable onto the column space of the design matrix, an elegant connection that motivates the use of projectors in linear models (see [5, 8]). Further applications in econometrics and prediction can be found in Stahlecker and Trenkler [9]. For a comprehensive abstract treatment of projectors in Hilbert spaces, the monograph of Berberian [2] and the paper of Friedrichs [4] are standard references. Yanai et al. [11] examined properties of projection matrices in finite-dimensional Euclidean spaces, noting that once a subspace $V \subset \mathbb{E}^n$ is specified, there are infinitely many complementary subspaces, but the orthogonal complement $V^\perp$ is unique. Vasilevski [10] explored orthogonal projections in the context of C^*-algebras, and listed several applications including Gram–Schmidt orthogonalization, QR decomposition, and orthogonal polynomials.

This paper provides a self-contained study of orthogonal projectors and their fundamental properties in Hilbert spaces. We begin by reviewing the basic properties of orthogonal complements and closed subspaces. We then prove the projection theorem, which states that for any closed subspace M of a Hilbert space H, every vector $x \in H$ admits a unique decomposition $x = y + z$ with $y \in M$ and $z \in M^\perp$; consequently $H = M \oplus M^\perp$ and $M^{\perp\perp} = M$. In addition, we discuss convex sets in Hilbert spaces and establish the existence and uniqueness of the best approximation from a closed convex subset, a result that generalizes the projection theorem. Finally, we characterize dense subspaces via orthogonality and study properties of orthogonal sums.

2 Basic Concepts and Preliminary Results

Throughout, H denotes a Hilbert space over the scalar field $\mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$), with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. For a sequence (x_n) in H, we write $x_n \to x$ to mean that (x_n) converges to x *in norm*, i.e. $\|x_n - x\| \to 0$; this is often called *strong convergence* to distinguish it from weak convergence. We first recall the matrix notion of a projector, which motivates the more general Hilbert space definitions used later in the paper.

Definition 2.1. *If $P \in \mathbb{C}^{m\times m}$ is a square matrix such that $P^2 = P$ then P is called a projector.*

Definition 2.2. *A matrix $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector if $P^2 = P$ and $P = P^T$.*

Definition 2.3. *Given an orthogonal projector $P \in \mathbb{R}^{n\times n}$, the associated complementary projector is $I - P$.*

The matrix notions above are the finite-dimensional instances of a more general idea: a bounded linear idempotent operator on a Hilbert space is an orthogonal projector precisely when it is self-adjoint. The next definitions extend the discussion to bounded operators on an arbitrary (possibly infinite-dimensional) Hilbert space H, which is the setting used for the rest of the paper.

Definition 2.4. *Let H be a Hilbert space and $T \in B(H)$. A closed linear subspace M of H is said to* reduce *T if both M and $M^\perp$ are T-invariant.*

Definition 2.5. *Let H be a Hilbert space and $T \in B(H)$. An orthogonal projector P is said to* reduce *T if $PT = TP$, i.e. P commutes with T. This is equivalent to saying that M reduces T, where $M = \mathfrak{R}_P$ is the range of P.*

Proposition 1. *Let $P : H \to H$ be a bounded linear map on the Hilbert space H such that $P^2 = P$. Then the following are equivalent:*

(i) P is self-adjoint;

(ii) P is normal;

(iii) $x - Px$ *is orthogonal to* Px *for every* $x \in H$*.*

If these conditions hold, P *is called the* orthogonal projection *onto its range.*

Proof. (i) $\Rightarrow$ (ii): If $P^* = P$ then $PP^* = P^2 = P = P^2 = P^*P$, so P is normal.

(ii) $\Rightarrow$ (iii): Recall that for any bounded operator T on H, $\|Tx\| = \|T^*x\|$ for all x if and only if T is normal, and consequently $\ker T = \ker T^*$ for normal T. Since P is idempotent, $x - Px \in \ker P$ for every x, because $P(x - Px) = Px - P^2x = Px - Px = 0$. Because P is bounded and idempotent, its range is closed and $\operatorname{ran}(P) = (\ker P^*)^\perp$; since P is normal, $\ker P^* = \ker P$, so $\operatorname{ran}(P) = (\ker P)^\perp$. As $Px \in \operatorname{ran}(P)$ and $x - Px \in \ker P$, orthogonality of $\operatorname{ran}(P)$ and $\ker P$ gives $\langle x - Px,\, Px\rangle = 0$, which is (iii).

(iii) $\Rightarrow$ (i): Assume $\langle x - Px, Px\rangle = 0$ for all $x \in H$, i.e.

$$\langle x, Px\rangle = \langle Px, Px\rangle \qquad \text{for all } x \in H. \tag{$*$}$$

Replacing x by $x + y$ in $(*)$ and expanding both sides using linearity of the inner product (in the first argument) and $P^2 = P$ yields, after simplification,

$$\langle x, Py\rangle + \langle y, Px\rangle = \langle Px, Py\rangle + \langle Py, Px\rangle \qquad \text{for all } x, y \in H.$$

Applying $(*)$ with x replaced by y shows $\langle y, Py\rangle = \langle Py, Py\rangle$, and a symmetric computation (swapping the roles of x and y, together with the sesquilinearity of the inner product) forces

$$\langle x, Py\rangle = \langle Px, y\rangle \qquad \text{for all } x, y \in H,$$

that is, $\langle Px, y\rangle = \langle x, Py\rangle = \overline{\langle Py, x\rangle} = \overline{\langle P^*x, y\rangle}$ is consistent precisely when $\langle P^*x, y\rangle = \langle Px, y\rangle$ for all x, y, i.e. $P^* = P$. Hence P is self-adjoint. □

3 Main Results

Theorem 3.1. *Let* H *be a Hilbert space and let* M, N *be nonempty subsets of* H*. Then:*

(i) $M^\perp$ *is a closed linear subspace of* H*;*

(ii) $M \cap M^\perp \subseteq \{0\}$ *(with equality if* M *is itself a linear subspace);*

(iii) $M \subseteq M^{\perp\perp}$*;*

(iv) $M \subseteq N \Rightarrow N^\perp \subseteq M^\perp$*;*

(v) $M^\perp = M^{\perp\perp\perp}$*.*

Proof. (i) By definition $M^\perp = \{x \in H : x \perp M\} = \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in M\}$.

Subspace. Let $y, z \in M^\perp$ and $\alpha, \beta \in \mathbb{K}$. For every $x \in M$,

$$\langle \alpha y + \beta z, x\rangle = \alpha\langle y, x\rangle + \beta\langle z, x\rangle = \alpha \cdot 0 + \beta \cdot 0 = 0,$$

so $\alpha y + \beta z \in M^\perp$. Hence $M^\perp$ is a linear subspace of H.

Closedness. Let $z \in \overline{M^\perp}$. Then there is a sequence (z_n) in $M^\perp$ with $z_n \to z$. Fix $x \in M$. By continuity of the inner product, $\langle z_n, x\rangle \to \langle z, x\rangle$. Since $\langle z_n, x\rangle = 0$ for every n, we get $\langle z, x\rangle = 0$. As $x \in M$ was arbitrary, $z \in M^\perp$. Thus $\overline{M^\perp} \subseteq M^\perp$; the reverse inclusion is automatic, so $M^\perp$ is closed.

(ii) If $x \in M \cap M^\perp$ then $x \perp M$ and $x \in M$, so $\langle x, x\rangle = 0$, i.e. $\|x\|^2 = 0$, so $x = 0$. Hence $M \cap M^\perp \subseteq \{0\}$. If M is a linear subspace then $0 \in M$, so $0 \in M \cap M^\perp$ and equality holds.

(iii) Let $x \in M$. We must show $x \perp M^\perp$, i.e. $x \in (M^\perp)^\perp = M^{\perp\perp}$. For any $y \in M^\perp$ we have $\langle y, x\rangle = 0$ (since $x \in M$), hence $\langle x, y\rangle = \overline{\langle y, x\rangle} = 0$. Thus $x \perp y$ for every $y \in M^\perp$, i.e. $x \in M^{\perp\perp}$.

(iv) Suppose $M \subseteq N$ and let $x \in N^\perp$, so $\langle x, y\rangle = 0$ for all $y \in N$. Since $M \subseteq N$, this holds in particular for all $y \in M$, so $x \perp M$, i.e. $x \in M^\perp$. Hence $N^\perp \subseteq M^\perp$.

(v) By (iii) applied to M, $M \subseteq M^{\perp\perp}$; applying (iv) to this inclusion gives $(M^{\perp\perp})^\perp \subseteq M^\perp$, i.e. $M^{\perp\perp\perp} \subseteq M^\perp$. On the other hand, (iii) applied to $M^\perp$ gives $M^\perp \subseteq (M^\perp)^{\perp\perp} = M^{\perp\perp\perp}$. Combining the two inclusions, $M^\perp = M^{\perp\perp\perp}$. □

Definition 3.1. *Let M, N be linear subspaces of H with $M \subseteq N$. We denote by $N \ominus M$ the set*

$$N \ominus M := \{y \in N : y \perp M\} = N \cap M^\perp,$$

called the orthogonal complement of M relative to N.

Lemma 1. *If N is a closed linear subspace of H and $M \subseteq N$ is a linear subspace of H, then $N \ominus M$ is a closed linear subspace of H.*

Proof. By definition, $N \ominus M = N \cap M^\perp$. Since N is closed by hypothesis and $M^\perp$ is always closed (Theorem 3.1(i)), and the intersection of two closed linear subspaces is again a closed linear subspace, $N \ominus M = N \cap M^\perp$ is a closed linear subspace of H. □

Lemma 2. *Let M be a nonempty subset of H, let $S = \operatorname{span} M$ denote its linear span, and let $\overline{S}$ denote the closure of S. Then*

$$M^\perp = S^\perp = \overline{S}^\perp.$$

Proof. Since $M \subseteq S \subseteq \overline{S}$, part (iv) of Theorem 3.1 gives

$$\overline{S}^\perp \subseteq S^\perp \subseteq M^\perp.$$

It remains to show $M^\perp \subseteq \overline{S}^\perp$. Let $x \in M^\perp$. Every element of S is a finite linear combination $\sum_i \alpha_i m_i$ of elements $m_i \in M$, so by linearity of the inner product,

$$\left\langle x, \sum_i \alpha_i m_i \right\rangle = \sum_i \alpha_i \langle x, m_i \rangle = 0,$$

hence $x \perp S$, i.e. $x \in S^\perp$. Now let $w \in \overline{S}$; choose $s_n \in S$ with $s_n \to w$. Since $\langle x, s_n \rangle = 0$ for all n, continuity of the inner product gives $\langle x, w \rangle = \lim_n \langle x, s_n \rangle = 0$. Hence $x \perp \overline{S}$, i.e. $x \in \overline{S}^\perp$. This proves $M^\perp \subseteq \overline{S}^\perp$, and combined with the first inclusion, $M^\perp = S^\perp = \overline{S}^\perp$. □

Definition 3.2. *Let X be a linear space over $\mathbb{K}$. A nonempty subset M of X is said to be* convex *if for every $x, y \in M$ and every $\lambda \in \mathbb{R}$ with $0 < \lambda < 1$, the element $\lambda x + (1-\lambda)y$ also belongs to M.*
Proposition 2. *Let X be a normed linear space over $\mathbb{K}$ and let M be a convex subset of X. Then $\overline{M}$ is also convex.*

Proof. Let $x, y \in \overline{M}$ and $0 < \lambda < 1$. Since $x, y \in \overline{M}$, there exist sequences (x_n) and (y_n) of elements of M such that $x_n \to x$ and $y_n \to y$. Since $x_n, y_n \in M$ and M is convex,

$$\lambda x_n + (1-\lambda) y_n \in M \qquad \text{for every } n.$$

Because $x_n \to x$ and $y_n \to y$, and vector space operations are continuous, $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$. As this is a limit of points of M, $\lambda x + (1-\lambda)y \in \overline{M}$. Hence $\overline{M}$ is convex. □

Theorem 3.2 (Best approximation theorem)**.** *Let H be a Hilbert space and let M be a nonempty closed convex subset of H (it suffices that M have the* midpoint property*: for every $y_1, y_2 \in M$, $\frac{1}{2}(y_1 + y_2) \in M$). Let $x \in H$ and let $r = \operatorname{dist}(x, M) = \inf\{\|y - x\| : y \in M\}$. Then there exists a unique $y_0 \in M$ such that $\|x - y_0\| = r$.*

Proof. **Existence.** If $x \in M$, take $y_0 = x$; then $r = 0 = \|x - y_0\|$ and we are done. Suppose now $r \geq 0$ is the infimum in general (this includes $r = 0$ with $x \notin M$, and $r > 0$). Since $r = \inf\{\|y - x\| : y \in M\}$, by the definition of infimum there is a sequence (y_n) in M with $\|y_n - x\| \to r$.

We show (y_n) is Cauchy. By the parallelogram law applied to $u = y_n - x$ and $v = y_m - x$,

$$\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

that is,

$$\|y_n - y_m\|^2 + \|(y_n + y_m) - 2x\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2,$$

so

$$\|y_n - y_m\|^2 = 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4\left\|\frac{y_n + y_m}{2} - x\right\|^2. \tag{3.1}$$

Since $y_n, y_m \in M$ and M has the midpoint property, $\frac{1}{2}(y_n + y_m) \in M$, so by definition of r as an infimum,

$$\left\|\frac{y_n + y_m}{2} - x\right\| \geq r.$$

Substituting into (3.1),

$$\|y_n - y_m\|^2 \leq 2\|y_n - x\|^2 + 2\|y_m - x\|^2 - 4r^2.$$

As $m, n \to \infty$, $\|y_n - x\| \to r$ and $\|y_m - x\| \to r$, so the right-hand side tends to $2r^2 + 2r^2 - 4r^2 = 0$. Hence $\|y_n - y_m\| \to 0$ as $m, n \to \infty$, i.e. (y_n) is a Cauchy sequence. Since H is complete, $y_n \to y_0$ for some $y_0 \in H$. As $y_n \in M$ for every n and M is closed, $y_0 \in M$. Finally, by continuity of the norm,

$$\|x - y_0\| = \lim_{n\to\infty} \|x - y_n\| = r.$$

Uniqueness. Suppose $y_0, y_0' \in M$ both satisfy $\|x - y_0\| = \|x - y_0'\| = r$. By the parallelogram law applied to $x - y_0$ and $x - y_0'$,

$$\|y_0 - y_0'\|^2 = 2\|x - y_0\|^2 + 2\|x - y_0'\|^2 - 4\left\|x - \frac{y_0 + y_0'}{2}\right\|^2.$$

Since $\frac{1}{2}(y_0 + y_0') \in M$ by the midpoint property, $\left\|x - \frac{1}{2}(y_0 + y_0')\right\| \geq r$, so

$$\|y_0 - y_0'\|^2 \leq 2r^2 + 2r^2 - 4r^2 = 0,$$

whence $y_0 = y_0'$. □

Corollary 1. *Let H be a Hilbert space and let N be a linear subspace of H that properly contains a closed linear subspace M of H. Then there is a nonzero vector $z \in N$ such that $z \perp M$.*

Proof. Since N properly contains M, there exists $x \in N \setminus M$. Because M is closed and $x \notin M$, $r := \operatorname{dist}(x, M) > 0$. By Theorem 3.2 there is a (unique) $y_0 \in M$ with $\|x - y_0\| = r$, and the argument in the proof of Theorem 3.3 below shows $x - y_0 \perp M$. Set $z = x - y_0$. Since $r \neq 0$, $z \neq 0$. Also $x \in N$ and $y_0 \in M \subseteq N$, so $z = x - y_0 \in N$ because N is a linear subspace. Hence $z \neq 0$, $z \in N$, and $z \perp M$. □

Corollary 2. *Let H be a Hilbert space, let N be a closed linear subspace of H, and let M be a linear subspace of H with $M \subseteq N$. Then $\overline{M} = N$ if and only if*

$$z \in N \text{ and } z \perp M \implies z = 0.$$

Proof. ($\Rightarrow$) Suppose $\overline{M} = N$, and let $z \in N$ with $z \perp M$. Since M is dense in N, for every $w \in N = \overline{M}$ there is a sequence (m_k) in M with $m_k \to w$; as $\langle z, m_k\rangle = 0$ for all k, continuity of the inner product gives $\langle z, w\rangle = 0$. Taking $w = z \in N$ gives $\langle z, z\rangle = 0$, i.e. $\|z\|^2 = 0$, so $z = 0$.

($\Leftarrow$) Suppose $z \in N,\ z \perp M \Rightarrow z = 0$. Since $M \subseteq N$ and N is closed, $\overline{M} \subseteq N$. Suppose, for contradiction, that $\overline{M} \neq N$; then $\overline{M}$ is a closed subspace properly contained in N. By Corollary 1 (applied with $\overline{M}$ in place of M), there is a nonzero $z \in N$ with $z \perp \overline{M}$. In particular $z \perp M$ (since $M \subseteq \overline{M}$), so $z \in N$ and $z \perp M$; by hypothesis this forces $z = 0$, contradicting $z \neq 0$. Hence $\overline{M} = N$. □

Corollary 3. *Let H be a Hilbert space and let M be a linear subspace of H. Then M is dense in H if and only if*

$$x \in H,\ x \perp M \implies x = 0.$$

Proof. This is the special case $N = H$ of Corollary 2, since H is trivially a closed subspace of itself. □

Definition 3.3. *Let X be a linear space and let M, N be linear subspaces of X. The* linear sum *of M and N is*

$$M + N := \{x + y : x \in M,\ y \in N\}.$$

Remark 1. $M + N$ is a linear subspace of X (in particular $M + N \subseteq X$). Indeed, let $u, v \in M + N$ and $\alpha, \beta \in \mathbb{K}$. Then there exist $x, x' \in M$ and $y, y' \in N$ with $u = x + y$ and $v = x' + y'$, so

$$\alpha u + \beta v = \alpha(x + y) + \beta(x' + y') = (\alpha x + \beta x') + (\alpha y + \beta y').$$

Since M, N are subspaces, $\alpha x + \beta x' \in M$ and $\alpha y + \beta y' \in N$, so $\alpha u + \beta v \in M + N$. Hence $M + N$ is a linear subspace of X.

Definition 3.4. *Let X be a linear space and let M, N be linear subspaces of X. If $M \cap N = \{0\}$, the linear sum $M + N$ is denoted $M \oplus N$ and called the (internal)* direct sum *of M and N.*

Lemma 3. *If X is a linear space and $X = M \oplus N$ for linear subspaces M, N of X, then each $x \in X$ can be decomposed uniquely as $x = y + z$ with $y \in M$ and $z \in N$.*

Proof. Existence follows from $X = M + N$. For uniqueness, suppose $x = y + z = y' + z'$ with $y, y' \in M$ and $z, z' \in N$. Then $y - y' = z' - z$. But $y - y' \in M$ and $z' - z \in N$, and $M \cap N = \{0\}$, so $y - y' = 0 = z' - z$, i.e. $y = y'$ and $z = z'$. □

Theorem 3.3 (Projection theorem)**.** *Let H be a Hilbert space and let M be a closed linear subspace of H. Then*

$$H = M \oplus M^{\perp}, \qquad \textit{and moreover} \qquad M^{\perp\perp} = M.$$

Proof. **Case** $M = H$**.** Here $M^{\perp} = H^{\perp} = \{0\}$ (if $x \in H^{\perp}$ then $\langle x, x\rangle = 0$, so $x = 0$). Every $x \in H$ has the trivial representation $x = x + 0$ with $x \in M = H$ and $0 \in M^{\perp}$, and $M \cap M^{\perp} = \{0\}$, so $H = M \oplus M^{\perp}$. Also $M^{\perp\perp} = \{0\}^{\perp} = H = M$.

Case $M \neq H$ (proper closed subspace). Since M is a closed convex subset of H (indeed a closed subspace, hence certainly has the midpoint property), Theorem 3.2 gives, for each $x \in H$, a unique $y_0 \in M$ minimizing $\|x - y\|$ over $y \in M$.

We claim $x - y_0 \in M^\perp$. Let $m \in M$ and $t \in \mathbb{R}$ be arbitrary. Since M is a subspace, $y_0 + tm \in M$, so by minimality of y_0,

$$\|x - y_0 - tm\|^2 \geq \|x - y_0\|^2.$$

Expanding the left side,

$$\|x - y_0\|^2 - 2t \operatorname{Re}\langle x - y_0, m\rangle + t^2\|m\|^2 \geq \|x - y_0\|^2,$$

so $-2t \operatorname{Re}\langle x - y_0, m\rangle + t^2\|m\|^2 \geq 0$ for all $t \in \mathbb{R}$. If $\operatorname{Re}\langle x - y_0, m\rangle \neq 0$, choosing t small and of the appropriate sign makes the left side negative, a contradiction. Hence $\operatorname{Re}\langle x - y_0, m\rangle = 0$ for every $m \in M$. If $\mathbb{K} = \mathbb{C}$, replacing m by im (which also lies in M) gives $\operatorname{Re}\langle x - y_0, im\rangle = \operatorname{Im}\langle x - y_0, m\rangle = 0$ as well, so $\langle x - y_0, m\rangle = 0$ for every $m \in M$. Thus $x - y_0 \in M^\perp$, proving the claim.

Writing $x = y_0 + (x - y_0)$ with $y_0 \in M$ and $x - y_0 \in M^\perp$ shows $H = M + M^\perp$. Since M is a subspace, $M \cap M^\perp = \{0\}$ by Theorem 3.1(ii), so by Lemma 3, $H = M \oplus M^\perp$.

Proof that $M^{\perp\perp} = M$. By Theorem 3.1(iii), $M \subseteq M^{\perp\perp}$. For the reverse inclusion, let $x \in M^{\perp\perp}$. By the decomposition just established, write $x = x' + x''$ with $x' \in M$ and $x'' \in M^\perp$. Since $M \subseteq M^{\perp\perp}$ and $M^{\perp\perp}$ is a linear subspace, $x'' = x - x' \in M^{\perp\perp}$ (as $x \in M^{\perp\perp}$ and $x' \in M \subseteq M^{\perp\perp}$). But also $x'' \in M^\perp$, so $x'' \in M^\perp \cap M^{\perp\perp} = \{0\}$ (Theorem 3.1(ii), applied to the subspace $M^\perp$). Hence $x'' = 0$, so $x = x' \in M$. Thus $M^{\perp\perp} \subseteq M$, and equality follows. □

Remark 2. Since $M^\perp$ is always a closed linear subspace of H (Theorem 3.1(i)), applying Theorem 3.3 to $M^\perp$ in place of M gives

$$H = M^\perp \oplus M^{\perp\perp}.$$

Proposition 3. *Let M be a closed linear subspace of a Hilbert space H. There is a unique linear subspace N of H such that $M \perp N$ and $H = M \oplus N$; namely $N = M^\perp$.*

Proof. **Existence.** $N = M^\perp$ satisfies $M \perp N$ trivially, and $H = M \oplus M^\perp$ by Theorem 3.3.

Uniqueness. Suppose N is any linear subspace with $M \perp N$ and $H = M \oplus N$. Since $N \perp M$, certainly $N \subseteq M^\perp$. For the reverse inclusion, let $x \in M^\perp$. Since $H = M \oplus N$, write $x = m + n$ uniquely with $m \in M$, $n \in N$. As $N \subseteq M^\perp$ and $x \in M^\perp$, both x and n lie in $M^\perp$, so $m = x - n \in M^\perp$ as well; but also $m \in M$, so $m \in M \cap M^\perp = \{0\}$, giving $m = 0$. Hence $x = n \in N$. Thus $M^\perp \subseteq N$, and combined with $N \subseteq M^\perp$ we get $N = M^\perp$. □

Theorem 3.4. *If M, N are closed linear subspaces of a Hilbert space H and $M \perp N$, then $M + N$ is also a closed linear subspace of H.*

Proof. $M + N$ is a linear subspace of H (shown above). Let $z \in \overline{M + N}$. Then there is a sequence (z_n) in $M + N$ with $z_n \to z$. For each n, write $z_n = x_n + y_n$ with $x_n \in M$, $y_n \in N$. Then for all $m, n \in \mathbb{N}$,

$$z_m - z_n = (x_m + y_m) - (x_n + y_n) = (x_m - x_n) + (y_m - y_n),$$

where $x_m - x_n \in M$ and $y_m - y_n \in N$. Since $M \perp N$, $x_m - x_n \perp y_m - y_n$, so by the Pythagorean theorem,

$$\|z_m - z_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2.$$

Since (z_n) converges, it is Cauchy, so the left side tends to 0 as $m, n \to \infty$; as both terms on the right are nonnegative, each must also tend to 0. Hence (x_n) and (y_n) are both Cauchy in H. Since H is complete, there exist $x, y \in H$ with $x_n \to x$ and $y_n \to y$. Because $x_n \in M$ for all n and M is closed, $x \in M$; similarly $y \in N$. Finally, since $z_n = x_n + y_n$ for all n and addition is continuous, $z = \lim_n z_n = x + y$. As $x \in M$ and $y \in N$, $z = x + y \in M + N$. This shows $\overline{M + N} \subseteq M + N$, so $M + N$ is closed. □

4 Conclusion

This study examined the theory of orthogonal projectors in Hilbert spaces and established several fundamental results that underpin functional analysis. The properties of orthogonal complements were investigated, and it was shown that the orthogonal complement of any subset of a Hilbert space is a closed linear subspace. The role of convexity in approximation theory was explored through the existence and uniqueness of best approximations from closed convex subsets.

A central result of the paper is the projection theorem, which shows that every vector in a Hilbert space can be uniquely represented as the sum of a vector in a closed subspace and a vector in its orthogonal complement. This decomposition yields the identity $M^{\perp\perp} = M$ for closed linear subspaces, and provides criteria for the density of a subspace in terms of orthogonality.

These results underline the importance of orthogonal projections as fundamental tools in Hilbert space theory, and they provide the theoretical basis for applications in linear algebra, approximation theory, optimization, signal processing, numerical analysis, and operator theory.

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