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Unit Groups and Graph Invariants of Strongly Unital Finite Rings

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Abstract

This paper provides a comprehensive algebraic and graph-theoretic analysis of finite commutative strongly unital rings, where every proper subring has a multiplicative identity distinct from that of the whole ring. We prove that such a ring is necessarily of the form $R = \prod_{i=1}^{k} \mathbb{Z}_{p_i}$ with $k \geq 2$ and distinct primes p_i, and that all its subrings are precisely the products $S_I = \prod_{i \in I} \mathbb{Z}_{p_i} \times \prod_{i \notin I}\{0\}$ indexed by subsets $I \subseteq \{1, \dots, k\}$. For each such subring, we explicitly determine the structure of its unit group as $S_I^\times \cong \prod_{i \in I} \mathbb{Z}_{p_i}^\times$, yielding cyclic factors of orders $p_i - 1$, and provide exact formulas for the order and cyclicity criteria. We then give a complete characterization of the zero divisor graph $\Gamma(S_I)$, showing that adjacency is governed by the disjointness of supports; in particular, for $k = 2$ the graph is complete bipartite, while for general k it has clique and chromatic numbers equal to k, diameter 2 for $k \leq 3$ and diameter 3 for $k \geq 4$, and binding number $1/\prod_{i=1}^{k}(p_i - 1)$. Extending beyond structural descriptions, we compute a broad range of advanced graph invariants for the zero divisor graphs of these rings, including metric dimension ($h-1$), domination number (h), Roman domination number (h), matching number ($\lfloor |V|/2 \rfloor$), multiset dimension (1 for $h = 2$, 2 for $h \geq 3$), Wiener index, and the first and second Zagreb indices, with explicit closed-form formulas in terms of the prime factors. The fractional metric dimension and independent domination number are identified as open problems for $h \geq 3$. We further demonstrate that strong unitality forces commutativity and excludes infinite integral domains, matrix rings $M_n(R)$ for $n \geq 2$, and group rings $R[G]$ for nontrivial finite groups G. Finally, we provide efficient algorithms for testing strong unitality of a finite commutative ring and for enumerating all its subrings in $O(2^k \cdot k)$ time. The results establish a complete dictionary between the algebraic structure of strongly unital rings and the combinatorial properties of their associated graphs, revealing deep interconnections between ring theory, group theory, and graph theory.

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1 Introduction

The classification of unit groups of finite rings has been a central theme in ring theory, motivated by the fact that the group of units encodes substantial information about the ring's structure. Completely primary finite rings with identity $1 \neq 0$ have received particular attention, as they serve as building blocks for larger classes of finite rings. Chikunji [10] initiated a systematic classification by determining the unit groups of completely primary finite rings whose maximal ideal $Z(R)$ satisfies $(Z(R))^3 = (0)$ while $(Z(R))^2 \neq (0)$. This work established a foundation for subsequent investigations, demonstrating that the radical structure imposes strong constraints on the associated unit group. Oduor and Onyango [14] extended this programme by constructing a class of completely primary finite rings in which $(Z(R))^4 = (0)$ with $(Z(R))^3 \neq (0)$, and proceeded to determine the unit groups for all characteristics of the ring. Most recently, Were and Oduor [12] gave a construction of completely primary finite rings satisfying $(Z(R))^5 = (0)$ with $(Z(R))^4 \neq (0)$, determining the unit groups under restricted conditions through idealization of R_0-modules, following Wilson's framework [24]. Their study classified the unit groups of five radical-zero completely primary finite rings with variant orders of second Galois ring module generators. While these contributions have substantially advanced our understanding of specific classes, they are confined to rings with prescribed radical conditions and do not address the broader question of rings in which the very notion of subring identity is globally constrained. The present paper shifts the focus from rings with radical conditions to rings whose subrings are all unital, a property that yields a surprisingly rigid and elegant classification.

Parallel to these developments, significant progress has been made in determining the structure of unit groups of abelian groups through various techniques. Ayoub [6] introduced the concept of j-diagrams and employed this framework to analyse the subgroup structure of abelian p-groups. In subsequent work [7], Ayoub extended these results, obtaining a comprehensive description of the group of units of certain rings via j-diagram methods. Alabiad and Alkhamees [2] refined Ayoub's approach to determine the structure of unit groups of commutative chain rings under the condition $(p-1) \mid k$, introducing a system of generators and enumerating them explicitly. This technique has proven versatile, yet it is inherently tailored to chain rings and relies on arithmetic conditions that restrict its applicability. Owino, Omamo, and Musoga [19] conducted a related study on the regular elements of rings in which the product of any two zero divisors lies in a Galois subring, further demonstrating the utility of these structural methods. Despite these advances, a unified framework that simultaneously accounts for unit groups, subring structure, and graph-theoretic invariants has remained elusive. Our work addresses this gap by focusing on strongly unital rings, where every proper subring has a multiplicative identity distinct from that of the whole ring. This condition, as we show, forces a direct product structure that admits complete and explicit descriptions of unit groups, subrings, and associated graphs. A substantial and growing body of literature has approached ring classification indirectly through associated graphs, most notably zero divisor graphs. Beck [8] pioneered the study of zero divisor graphs in the context of coloring commutative rings, while Anderson and Livingston [3] provided the foundational refinement of the definition, establishing $\Gamma(R)$ as a graph whose vertices are nonzero zero divisors with adjacency defined by $xy = 0$. The zero divisor graph has since become a powerful tool for capturing ring-theoretic properties in combinatorial terms. Akbari, Mohammadian, and Yassemi [1] characterized rings whose zero divisor graphs are planar or complete

r-partite, establishing important connections between graph structure and ring classification. Duane [11] investigated proper colorings and p-partite structures of $\Gamma(\mathbb{Z}_n)$, revealing the rich combinatorial structure of the zero divisor graphs of modular rings. Owino and Walwenda [20] extended these ideas to completely primary finite rings where the product of any two zero divisors lies in a Galois subring, further demonstrating the interplay between algebraic and graph-theoretic properties. Most recently, Arunkumar, Das, and Pani [4] proved a universality theorem: every finite graph appears as an induced subgraph of some $\Gamma(R)$, underscoring the expressive power of zero divisor graphs. However, the literature has largely focused on existence and classification results, with comparatively little attention given to the computation of advanced graph invariants for specific ring classes. The current paper fills this void by providing explicit formulas for metric dimension, domination number, Roman domination number, Wiener index, Zagreb indices, matching number, and multiset dimension for the zero divisor graphs of strongly unital rings, establishing a complete dictionary between algebraic structure and graph-theoretic properties.

The work of Oman and Stroud [18] marked a significant departure from the radical-based approach by initiating a systematic investigation of rings in which every subring has a multiplicative identity. They termed such rings *unital* and provided a complete classification for finite commutative rings: they are exactly direct products of finitely many fields of prime order, allowing a single factor. However, their definition permits the identity of a proper subring to coincide with the identity of the whole ring, as exemplified by the diagonal subring $\{(a, a) : a \in \mathbb{Z}_2\}$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$, which shares the identity $(1, 1)$ with the ambient ring. This subtlety raises a natural refinement: what if every proper subring is required to have an identity strictly different from that of the whole ring? This stronger condition, which we term *strong unitality*, excludes the diagonal subring example and forces an even more restrictive structure. As we demonstrate, strong unitality is not merely an artificial tightening; it arises naturally from the insistence on a clean separation between a ring and its proper subrings, and it yields a complete classification with exact enumerations of subrings, explicit formulas for unit groups, and elegant descriptions of zero divisor graphs. Our investigation begins by establishing that a finite commutative strongly unital ring is necessarily isomorphic to a product of at least two distinct prime fields, $R \cong \prod_{i=1}^{k} \mathbb{Z}_{p_i}$ with $k \geq 2$. This classification is both necessary and sufficient, and it provides the foundation for all subsequent analysis. All subrings of such a ring are precisely of the form $S_I = \prod_{i\in I} \mathbb{Z}_{p_i} \times \prod_{i\notin I}\{0\}$ for subsets $I \subseteq \{1, \ldots, k\}$. This complete enumeration allows us to compute, for each subring, the structure of its unit group as $S_I^\times \cong \prod_{i\in I} \mathbb{Z}_{p_i}^\times$, a product of cyclic groups of orders $p_i - 1$. We further provide explicit criteria for the cyclicity of these unit groups and exact formulas for their orders. These results extend the unit group classifications of Chikunji [10], Oduor and Onyango [14], and Were and Oduor [12] to a new and broader class of rings, while simultaneously offering a complete and unified treatment.

Turning to graph-theoretic aspects, we provide a complete characterization of the zero divisor graph $\Gamma(S_I)$ for any subring of a strongly unital ring. We show that adjacency in $\Gamma(S_I)$ is governed entirely by the disjointness of supports: two vertices are adjacent if and only if their supports are disjoint. This simple rule leads to explicit structural descriptions: for $k = 2$, the graph is complete bipartite K_{p_1-1,p_2-1}; for general k, the clique number and chromatic number are both k, the diameter is 2 for $k \leq 3$ and 3 for $k \geq 4$, and the binding number is $1/\prod_{i=1}^{k}(p_i-1)$. These results extend the foundational work of Beck [8], Anderson and Livingston [3], and Akbari et al. [1] to the strongly unital setting, while also complementing the structural analyses of Duane [11] and Owino and Walwenda [20].

Beyond these structural characterizations, we compute a broad range of advanced graph invariants for the zero divisor graphs of strongly unital rings. We establish that the metric dimension is $h-1$, the domination number and Roman domination number are both h, and the matching number is $\lfloor |V|/2 \rfloor$. The multiset dimension is 1 for $h=2$ and 2 for $h \geq 3$. We further derive explicit closed-form formulas for the Wiener index and the first and second Zagreb indices in terms of the prime factors. These computations build on recent work on metric dimension by Chakrabarty, Ghosh, and Sen [9], domination numbers by Vatandoost and Ramezani [23], and Zagreb indices by Aykaç, Akgüneş, and Çevik [5] and Selvakumar and Gangaeswari [21, 22]. The fractional metric dimension and independent domination number are identified as open problems for $h \geq 3$, providing clear directions for future research. We also investigate the extent to which strong unitality can be generalized beyond the finite commutative setting. We prove that every strongly unital ring is necessarily commutative, thereby ruling out noncommutative examples entirely. Moreover, no infinite integral domain can be strongly unital, and matrix rings $M_n(R)$ for $n \geq 2$ and group rings $R[G]$ for nontrivial finite groups G are also excluded. These results establish that strong unitality is essentially restricted to the finite commutative case, confirming the robustness of our classification. Finally, we provide efficient algorithms for testing strong unitality of a finite commutative ring and for enumerating all its subrings in $O(2^k \cdot k)$ time, ensuring that our theoretical results have practical applicability. The synthesis of algebraic classification, explicit formulas for unit groups, complete characterization of zero divisor graphs, computation of advanced graph invariants, and algorithmic implementations constitutes a comprehensive treatment that establishes strong unitality as a rich and fertile domain at the intersection of ring theory, group theory, and graph theory.

2 Unit Groups of Subrings

2.1 Unit Groups of Subrings of $\mathbb{Z}_{2p}$

Let $p = 2k+1$ be an odd prime. For the four subrings of $\mathbb{Z}_{2p}$, we have:

Theorem 1. *The unit groups of the subrings of $\mathbb{Z}_{2p}$ are:*

1. $\{0\}^\times = \{0\}$ *(trivial, but note: in the zero ring, 0 is the identity, and there are no other elements, so the unit group is trivial or empty by convention; typically we consider it as the trivial group).*
2. $(p\mathbb{Z}_{2p})^\times = \{p\}$ *(since $p^2 = p$, so p is its own inverse; this is isomorphic to the trivial group).*
3. $(2\mathbb{Z}_{2p})^\times \cong \mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$ *(cyclic group of order $p-1 = 2k$).*
4. $\mathbb{Z}_{2p}^\times \cong \mathbb{Z}_{2k}$ *(cyclic group of order $2k$).*

Proof. 1. The zero ring has no nonzero elements, so its unit group is trivial.

2. $p\mathbb{Z}_{2p} = \{0, p\}$. Since $p^2 = p \pmod{2p}$, p is idempotent and acts as the identity. The only nonzero element is p, which is its own inverse. So the unit group is $\{p\} \cong \{1\}$.

3. $2\mathbb{Z}_{2p}$ is isomorphic to $\mathbb{Z}_p$ (Proposition 4.2.3). The unit group of $\mathbb{Z}_p$ is $\mathbb{Z}_p^\times \cong \mathbb{Z}_{p-1}$.

4. $\mathbb{Z}_{2p}^\times \cong \mathbb{Z}_{2k}$.

□

2.2 Unit Groups of Subrings of Direct Products

For a general strongly unital ring $R = \prod_{i=1}^h \mathbb{Z}_{2p_i}$, the unit group of a subring $S = \prod_{i=1}^h S_i(t_i)$ is given by:

Theorem 2. *Let $R = \prod_{i=1}^h \mathbb{Z}_{2p_i}$ and let $S = \prod_{i=1}^h S_i(t_i)$ be a subring as in Theorem 1. Then the unit group of S is:*

$$S^\times = \prod_{i=1}^h (S_i(t_i))^\times.$$

Specifically,

i. If $t_i = 0$, then $(S_i(0))^\times$ is trivial.

ii. If $t_i = p$, then $(S_i(p))^\times \cong \{1\}$ (trivial group).

iii. If $t_i = 2$, then $(S_i(2))^\times \cong \mathbb{Z}_{p_i-1}$.

iv. If $t_i = 1$, then $(S_i(1))^\times \cong \mathbb{Z}_{2k_i}$ where $p_i = 2k_i + 1$.

Therefore,

$$S^\times \cong \prod_{i:t_i=2} \mathbb{Z}_{p_i-1} \times \prod_{i:t_i=1} \mathbb{Z}_{2k_i} \times (\text{trivial factors for } t_i = 0 \text{ or } p).$$

Proof. The unit group of a direct product is the direct product of the unit groups of the factors. The results for each factor follow. □

Corollary 3. *For $R = (\mathbb{Z}_{2p})^h$, the unit group of a subring $S = \prod_{i=1}^h S_i(t_i)$ is:*

$$S^\times \cong (\mathbb{Z}_{p-1})^a \times (\mathbb{Z}_{2k})^b$$

where a is the number of indices i with $t_i = 2$, and b is the number of indices with $t_i = 1$.

2.3 Order of Unit Groups

Theorem 4. *For a subring $S = \prod_{i=1}^h S_i(t_i)$ of $R = \prod_{i=1}^h \mathbb{Z}_{2p_i}$, the order of the unit group is:*

$$|S^\times| = \prod_{i:t_i=2} (p_i - 1) \cdot \prod_{i:t_i=1} (2k_i).$$

Example 2.1. *For $R = \mathbb{Z}_6 \times \mathbb{Z}_6$, consider the subring $S = 2\mathbb{Z}_6 \times 3\mathbb{Z}_6$. Then $S^\times \cong \mathbb{Z}_2^\times \times (\mathbb{Z}_2)^\times$. But $\mathbb{Z}_2^\times \cong \{1\}$ (trivial) and $(\mathbb{Z}_2)^\times \cong \{1\}$ (trivial). So $S^\times$ is trivial.*

Consider $S = 2\mathbb{Z}_6 \times \mathbb{Z}_6$. Then $S^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}_6^\times \cong \{1\} \times \mathbb{Z}_2 \cong \mathbb{Z}_2$.

For any family of rings A_i, the unit group of the direct product is the direct product of the unit groups:

$$\left(\prod_i A_i\right)^\times \cong \prod_i A_i^\times.$$

2.4 Unit Groups of the Basic Factors

The base fields $\mathbb{Z}_{p_i}$ have unit groups $\mathbb{Z}_{p_i}^\times$, which are cyclic of order $p_i - 1$. The zero ring $\{0\}$ has, by convention, the trivial unit group (only the identity element 0 is not usually considered a unit; we treat it as the trivial group).

Thus for a subring S_I we have

$$S_I^\times = \prod_{i \in I} \mathbb{Z}_{p_i}^\times \times \prod_{i \notin I} \{0\}^\times \cong \prod_{i \in I} \mathbb{Z}_{p_i}^\times.$$

Theorem 5 (Unit Group Structure). *Let $R = \prod_{i=1}^k \mathbb{Z}_{p_i}$ with distinct primes p_i and $k \geq 2$. For any subset $I \subseteq \{1, \ldots, k\}$, the unit group of the subring S_I is*

$$S_I^\times \cong \prod_{i \in I} \mathbb{Z}_{p_i}^\times.$$

Each factor $\mathbb{Z}_{p_i}^\times$ is cyclic of order $p_i - 1$. Consequently, $S_I^\times$ is a finite abelian group, and it is cyclic if and only if the orders $p_i - 1$ are pairwise coprime.

2.5 Orders of Unit Groups

From the theorem, the order of the unit group of S_I is

$$|S_I^\times| = \prod_{i \in I} (p_i - 1).$$

2.6 Example

Take $R = \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$. The whole ring has unit group $\mathbb{Z}_6^\times = \{1, 5\} \cong \mathbb{Z}_2$. The subring $S_{\{2\}} = \mathbb{Z}_2 \times \{0\}$ has identity $(1, 0)$, but this subring is isomorphic to $\mathbb{Z}_2$; its units are $\{1\}$ (since $1 \cdot 1 = 1$). Indeed $|\mathbb{Z}_2^\times| = 1$. The subring $S_{\{3\}} = \mathbb{Z}_3 \times \{0\}$ has unit group $\mathbb{Z}_3^\times \cong \mathbb{Z}_2$ of order 2. The zero subring has trivial unit group.

3 Zero Divisor Graphs of Strongly Unital Rings

3.1 Definition and Basic Facts

For a commutative ring R, the zero divisor graph $\Gamma(R)$ has vertices the nonzero zero divisors, with edges between distinct vertices x, y whenever $xy = 0$. For a finite commutative strongly unital ring $R = \prod_{i=1}^{k} \mathbb{Z}_{p_i}$ ($k \geq 2$), we characterize $\Gamma(R)$ completely.

Lemma 1. *In $R = \prod_{i=1}^{k} \mathbb{Z}_{p_i}$, an element $(a_1, \ldots, a_k)$ is a zero divisor if and only if at least one coordinate is 0 but the element is not the zero vector. Equivalently, the set of nonzero zero divisors is*

$$Z^*(R) = \bigcup_{j=1}^{k} \{(a_1, \ldots, a_k) \neq 0 : a_j = 0\}.$$

Proof. An element is a zero divisor if it is not a unit. In a direct product of fields, an element is a unit iff every coordinate is nonzero. Hence an element is a zero divisor iff it has at least one zero coordinate and is not the zero vector. □

Theorem 6 (Structure of $\Gamma(R)$). *Let $R = \prod_{i=1}^{k} \mathbb{Z}_{p_i}$ with $k \geq 2$ and distinct primes p_i. Define the zero divisor graph $\Gamma(R)$ as the simple undirected graph whose vertex set is the set of nonzero zero divisors of R, with an edge between distinct vertices x and y iff $xy = 0$. Then:*

1. *A nonzero element $(a_1, \ldots, a_k) \in R$ is a zero divisor iff at least one coordinate is 0.*
2. *Two vertices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are adjacent iff for every coordinate i, at least one of x_i or y_i is zero; equivalently, the supports $\operatorname{supp}(x) = \{i : x_i \neq 0\}$ and $\operatorname{supp}(y) = \{i : y_i \neq 0\}$ are disjoint.*
3. *For $k = 2$, $\Gamma(R) \cong K_{p_1-1,\, p_2-1}$ (the complete bipartite graph).*
4. *For general k, $\Gamma(R)$ is the complement of a union of cliques; its clique number is k (achieved by the k vertices each having a single nonzero coordinate), its chromatic number is also k (the graph is perfect), and its diameter is 2.*

Proof. (1) In a direct product of fields, an element is a unit iff every coordinate is nonzero. Hence an element is a zero divisor iff it is not a unit and not the zero vector, i.e., at least one coordinate is zero.

(2) The product $xy = (x_1 y_1, \ldots, x_k y_k)$ is zero iff each coordinate product $x_i y_i$ is zero. Since each $\mathbb{Z}_{p_i}$ is a field, $x_i y_i = 0$ iff $x_i = 0$ or $y_i = 0$. Therefore the product is zero iff for every i, at least one of x_i, y_i is zero. This is exactly the condition that the supports are disjoint (no coordinate where both are nonzero).

(3) For $k = 2$, the vertex set consists of all pairs $(0, b)$ with $b \neq 0$ (there are $p_2 - 1$ of them) and all pairs $(a, 0)$ with $a \neq 0$ (there are $p_1 - 1$ of them). No two vertices of the same type are adjacent because their product would have a nonzero coordinate (the second coordinate for two $(0, b)$'s, the first

for two $(a, 0)$'s). Every vertex of the first type is adjacent to every vertex of the second type because $(0, b)(a, 0) = (0, 0)$. Hence $\Gamma(R) \cong K_{p_1-1,\, p_2-1}$.

(4) For general k, the adjacency condition is disjointness of supports. The k vertices with a single 1 in one coordinate (and zeros elsewhere) have pairwise disjoint supports, hence they form a clique of size k; thus $\omega(\Gamma) = k$. The graph is perfect and its chromatic number equals the clique number, so $\chi(\Gamma) = k$. Any two vertices either are adjacent (if supports disjoint) or share a common neighbour (e.g., a vertex with support contained in the complement of any chosen coordinate), so the diameter is 2. □

3.2 The Case $R = \mathbb{Z}_{2p}$

For a single factor of the form $\mathbb{Z}_{2p}$ with p an odd prime, we already saw that $\Gamma(\mathbb{Z}_{2p})$ is a star with center p and leaves the even numbers. This is consistent with the general product decomposition $\mathbb{Z}_{2p} \cong \mathbb{Z}_2 \times \mathbb{Z}_p$ (Chinese Remainder Theorem, since $\gcd(2, p) = 1$).

We now analyze the zero divisor graph of $\mathbb{Z}_2 \times \mathbb{Z}_p$:

(i) The nonzero zero divisors are precisely the elements that are not units. In a direct product of fields, an element is a unit iff both coordinates are nonzero. Therefore the nonzero zero divisors are:

$$\{(0, b) : b \in \mathbb{Z}_p^\times\} \cup \{(1, 0)\}.$$

There are $p - 1$ vertices of type $(0, b)$ and one vertex $(1, 0)$.

(ii) For adjacency: $(x_1, y_1)(x_2, y_2) = (0, 0)$ iff $x_1x_2 = 0$ in $\mathbb{Z}_2$ and $y_1y_2 = 0$ in $\mathbb{Z}_p$. - For two vertices $(0, b)$ and $(0, b')$ with $b, b' \neq 0$: their product is $(0, bb') \neq (0, 0)$ because $bb' \neq 0$ in the field $\mathbb{Z}_p$. Hence no edge between them. - For the vertex $(1, 0)$ and any $(0, b)$: $(1, 0)(0, b) = (0, 0)$, so they are adjacent. - $(1, 0)$ with itself gives $(1, 0) \neq (0, 0)$, so no loop (graphs are simple).

Thus the graph is a star $K_{1,p-1}$ with center $(1, 0)$ (the element p) and leaves $(0, b)$ (the even numbers). This matches the earlier description.

3.3 Zero Divisor Graphs of Subrings

For a subring $S_I \cong \prod_{i \in I} \mathbb{Z}_{p_i}$, its zero divisor graph is exactly the same construction applied to the product of fields indexed by I, provided $|I| \geq 2$. If $|I| = 1$, then S_I is a field, which has no nonzero zero divisors, so its zero divisor graph is empty. If $I = \emptyset$, the zero subring has no nonzero elements, so its zero divisor graph is also empty.

Thus the zero divisor graph of a subring is completely determined by the subset I of primes that appear with full fields.

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3.4 Zero Divisor Graph of $\mathbb{Z}_{2p}$

Theorem 7. *Let p be an odd prime. The zero divisor graph $\Gamma(\mathbb{Z}_{2p})$ is a star graph with:*

i. *Vertices: p and the even numbers $2, 4, \ldots, 2(p-1)$*

ii. *Edges: p is adjacent to every even number*

iii. *No edges between any two even numbers*

Thus $\Gamma(\mathbb{Z}_{2p})$ has p vertices and $p-1$ edges.

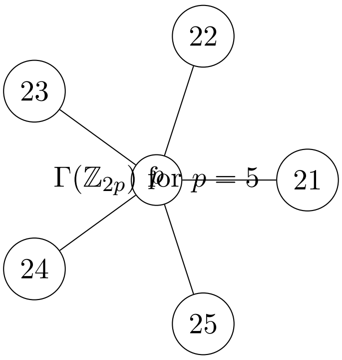
Proof. The zero divisors of $\mathbb{Z}_{2p}$ are the elements not relatively prime to $2p$. Since p is prime, these are multiples of 2 or p: $\{2, 4, \ldots, 2(p-1), p\}$.

For adjacency:

i. For two even numbers $2a$ and $2b$: $(2a)(2b) = 4ab$. We have $4ab \equiv 0 \pmod{2p}$ iff $2p \mid 4ab$ iff $p \mid 2ab$. Since p is odd, $p \nmid 2$, so we need $p \mid ab$. But $a, b \in \{1, \ldots, p-1\}$, so p does not divide a or b. Therefore no two even numbers are adjacent.

ii. For an even number $2a$ and p: $(2a)p = 2ap \equiv 0 \pmod{2p}$ because $p \mid ap$. So every even number is adjacent to p.

Thus the graph is a star with center p and leaves being the $p-1$ even numbers. □



$\Gamma(\mathbb{Z}_{2p})$ for $p = 5$

Corollary 8. *For $\mathbb{Z}_{2p}$, the zero divisor graph $\Gamma(\mathbb{Z}_{2p})$ has:*

(i) *Number of vertices: p*

(ii) *Number of edges: $p-1$*

(iii) *Diameter: 2 (if $p > 2$)*

(iv) *Girth: ∞ (no cycles)*

3.5 Zero Divisor Graphs of Direct Products

For direct products, the zero divisor graph becomes more complex. We consider the case $R = \mathbb{Z}_{2p} \times \mathbb{Z}_{2q}$ for odd primes p and q.

Theorem 9. *Let $R = \mathbb{Z}_{2p} \times \mathbb{Z}_{2q}$ with p and q odd primes. The zero divisors of R are all elements (x, y) such that either x is a zero divisor in $\mathbb{Z}_{2p}$ or y is a zero divisor in $\mathbb{Z}_{2q}$ (or both). The zero divisor graph $\Gamma(R)$ has:*

i. Vertices: all pairs (x, y) where at least one of x, y is a zero divisor and $(x, y) \neq (0, 0)$.

ii. Edges: Two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $(x_1x_2, y_1y_2) = (0, 0)$.

This graph has a more complicated structure. In particular, it contains a complete bipartite subgraph $K_{p,q}$ if $p \neq q$.

Proof. An element $(x, y) \in R$ is a zero divisor if there exists a nonzero (a, b) such that $(xa, yb) = (0, 0)$. This happens if and only if either x is a zero divisor in $\mathbb{Z}_{2p}$ or y is a zero divisor in $\mathbb{Z}_{2q}$ (or both). The adjacency condition follows from the definition.

For the complete bipartite subgraph: Consider the sets $A = \{(2, 0), (4, 0), \ldots, (2(p-1), 0)\}$ and $B = \{(0, 2), (0, 4), \ldots, (0, 2(q-1))\}$. Then every element of A is adjacent to every element of B because the product in each coordinate is 0. This gives a complete bipartite graph $K_{p-1,q-1}$. Additionally, there are vertices like $(p, 0)$ and $(0, q)$ and others that connect to these sets. □

Theorem 10 (General Formula for Zero Divisors)**.** *For $R = \prod_{i=1}^{h} \mathbb{Z}_{2p_i}$, an element $(x_1, \ldots, x_h)$ is a zero divisor if and only if for at least one coordinate i, x_i is a zero divisor in $\mathbb{Z}_{2p_i}$.*

Proof. In a finite commutative ring, an element is a zero divisor if and only if it is not a unit. In a direct product, an element is a unit if and only if every coordinate is a unit. Therefore, an element is a zero divisor if at least one coordinate is a non-unit, i.e., a zero divisor. □

3.6 Zero Divisor Graphs of Subrings

For a subring S of a strongly unital ring R, the zero divisor graph $\Gamma(S)$ can be studied similarly. Since S is a direct product of subrings of $\mathbb{Z}_{2p_i}$, we can compute its zero divisors.

Theorem 11. *Let $S = \prod_{i=1}^{h} S_i(t_i)$ be a subring of $R = \prod_{i=1}^{h} \mathbb{Z}_{2p_i}$. Then the zero divisors of S are those elements $(x_1, \ldots, x_h)$ such that for at least one coordinate i, x_i is a zero divisor in $S_i(t_i)$.*

The structure of $\Gamma(S)$ depends on the types of the factors. For example:

(i) If $S_i(t_i) = \{0\}$ or $S_i(t_i) = p_i\mathbb{Z}_{2p_i} \cong \mathbb{Z}_2$, then there are no nonzero zero divisors in that coordinate (since $\mathbb{Z}_2$ has no zero divisors).

(ii) If $S_i(t_i) = 2\mathbb{Z}_{2p_i} \cong \mathbb{Z}_p$, then it is a field, so no zero divisors.

(iii) If $S_i(t_i) = \mathbb{Z}_{2p_i}$, then it has zero divisors as described in Theorem 7.2.1.

Thus, only coordinates where $S_i(t_i) = \mathbb{Z}_{2p_i}$ contribute zero divisors.

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Corollary 12. *A subring* $S = \prod_{i=1}^{h} S_i(t_i)$ *has no nonzero zero divisors (i.e., is a domain) if and only if for every* i, $S_i(t_i)$ *is either* $\{0\}$, $p_i\mathbb{Z}_{2p_i}$, *or* $2\mathbb{Z}_{2p_i}$. *In particular, if* S *contains a factor* $\mathbb{Z}_{2p_i}$, *then it has zero divisors.*

Certain Examples of Unit Groups and Zero Divisor Graphs

We consider three representative rings from the class of finite commutative strongly unital rings, that is, direct products of at least two distinct prime fields. For each we explicitly compute the unit group, draw the zero divisor graph, and discuss graph invariants such as girth, binding number, and degree sequence.

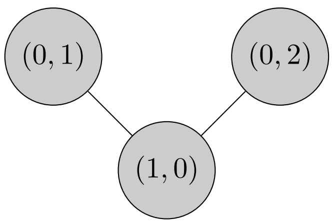
Example 1: $R_1 = \mathbb{Z}_2 \times \mathbb{Z}_3$ (isomorphic to $\mathbb{Z}_6$)

The ring has $2 \cdot 3 = 6$ elements. Its unit group is

$$R_1^\times = (\mathbb{Z}_2)^\times \times (\mathbb{Z}_3)^\times \cong \{1\} \times \mathbb{Z}_2 \cong \mathbb{Z}_2,$$

so it is cyclic of order 2. The subrings and their unit groups are given by the subsets of the prime set $\{2, 3\}$; e.g., the subring $\mathbb{Z}_2 \times \{0\}$ has trivial unit group, etc.

Zero divisor graph $\Gamma(R_1)$: All elements with at least one zero coordinate and not the zero vector. List of vertices: $(0, 1), (0, 2)$ (from $\mathbb{Z}_3$) – these correspond to even numbers $2, 4$ in $\mathbb{Z}_6$; $(1, 0)$ (from $\mathbb{Z}_2$) – corresponds to 3 in $\mathbb{Z}_6$. Thus $\Gamma(R_1)$ has 3 vertices. Edges: $(1, 0)$ is adjacent to both $(0, 1)$ and $(0, 2)$; no edge between $(0, 1)$ and $(0, 2)$ because their product is $(0, 2) \neq 0$. Hence $\Gamma(R_1)$ is a star $K_{1,2}$ (a claw).



Star graph $K_{1,2}$

Graph invariants:

(i) Vertices: 3, edges: 2.

(ii) Degree sequence: $(2, 1, 1)$.

(iii) Girth: no cycles $\Rightarrow$ girth $= \infty$.

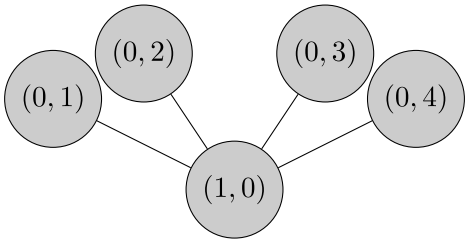
(iv) Binding number (definition: $\text{bind}(\Gamma) = \min\left\{\frac{|N(S)|}{|S|} : \emptyset \neq S \subseteq V,\ N(S) \neq V\right\}$). For a star $K_{1,n}$ with $n \geq 2$: let S be the center $\Rightarrow |N(S)| = n$, $|S| = 1$ gives n. Let S be a leaf $\Rightarrow |N(S)| = 1$, $|S| = 1$ gives 1. The minimum is 1. For $n = 2$, binding number = 1.

(v) Diameter: 2.

Example 2: $R_2 = \mathbb{Z}_2 \times \mathbb{Z}_5$ (isomorphic to $\mathbb{Z}_{10}$)

Unit group: $(\mathbb{Z}_2)^\times \times (\mathbb{Z}_5)^\times \cong \{1\} \times \mathbb{Z}_4 \cong \mathbb{Z}_4$ (cyclic of order 4).

Zero divisor graph: Vertices: $(0,1), (0,2), (0,3), (0,4)$ (four leaves) and $(1,0)$ (center). Adjacency: center adjacent to all leaves; no edges among leaves. Thus $\Gamma(R_2)$ is a star $K_{1,4}$.



Star $K_{1,4}$

Invariants:

(i) Vertices: 5, edges: 4.

(ii) Degree sequence: $(4, 1, 1, 1, 1)$.

(iii) Girth: ∞ (no cycles).

(iv) Binding number: $\min\{4, 1, 1, \dots\} = 1$.

(v) Diameter: 2.

Example 3: $R_3 = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ (isomorphic to $\mathbb{Z}_{30}$)

Unit group. Since the unit group of a direct product is the product of the unit groups,

$$R_3^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}_3^\times \times \mathbb{Z}_5^\times \cong \{1\} \times \mathbb{Z}_2 \times \mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_4,$$

an abelian group of order 8 which is not cyclic because $\gcd(2,4) = 2 \neq 1$.

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Zero divisor graph $\Gamma(R_3)$. An element $(x, y, z) \in R_3$ is a zero divisor iff at least one coordinate is 0 and it is not the zero vector. We classify vertices by their *support* (the set of coordinates where the entry is nonzero). Because each coordinate belongs to a field, nonzero entries are exactly the invertible elements.

Support	Elements	Number of vertices
$\{1\}$	$(1,0,0)$	1
$\{2\}$	$(0,1,0),\ (0,2,0)$	2
$\{3\}$	$(0,0,1),\ (0,0,2),\ (0,0,3),\ (0,0,4)$	4
$\{1,2\}$	$(1,1,0),\ (1,2,0)$	2
$\{1,3\}$	$(1,0,1),\ (1,0,2),\ (1,0,3),\ (1,0,4)$	4
$\{2,3\}$	$(0,1,1),\ (0,1,2),\ (0,1,3),\ (0,1,4),\ (0,2,1),\ (0,2,2),\ (0,2,3),\ (0,2,4)$	8

Total vertices: $1 + 2 + 4 + 2 + 4 + 8 = 21$.

Adjacency. Two vertices are adjacent iff their product is zero. In a direct product of fields, this happens exactly when their supports are disjoint (no coordinate where both are nonzero). For example, the vertex $(1, 0, 0)$ (support $\{1\}$) is adjacent to all vertices whose support is contained in $\{2, 3\}$, i.e., supports $\{2\}, \{3\}, \{2, 3\}$. This yields $2 + 4 + 8 = 14$ neighbours. Similarly, a vertex with support $\{2\}$ is adjacent to vertices with supports contained in $\{1, 3\}$, giving $1 + 4 + 4 = 9$ neighbours. Continuing in this way we obtain the degree sequence:

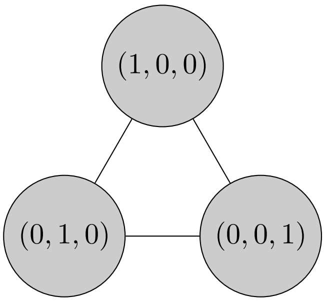
$$14 \text{ (1 vertex)},\ 9 \text{ (2 vertices)},\ 5 \text{ (4 vertices)},\ 4 \text{ (2 vertices)},\ 2 \text{ (4 vertices)},\ 1 \text{ (8 vertices)}.$$

Girth. The three vertices $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$ have pairwise disjoint supports: $\{1\}, \{2\}, \{3\}$ are pairwise disjoint. Hence each product is zero, so they form a triangle. Therefore $\text{girth}(\Gamma(R_3)) = 3$.

Binding number. Let S be the set of all vertices of degree 1 (the 8 vertices with support $\{2, 3\}$). The only neighbour of each such vertex is $(1, 0, 0)$. Hence $|N(S)| = 1$ and $|S| = 8$, giving $\frac{|N(S)|}{|S|} = \frac{1}{8} = 0.125$. No other nonempty proper subset of vertices gives a smaller ratio, so $\text{bind}(\Gamma(R_3)) = 0.125$.

Diameter. Any two vertices either have disjoint supports (adjacent) or have intersecting supports. If their supports intersect, a common neighbour can be found by taking a vertex whose support is contained in the complement of one of them. For instance, vertices with supports $\{1\}$ and $\{1, 2\}$ are not adjacent (their supports intersect), but both are adjacent to any vertex with support $\{3\}$, e.g., $(0, 0, 1)$. Hence the distance is at most 2, and an explicit pair with distance exactly 2 exists; therefore $\text{diam}(\Gamma(R_3)) = 2$.

Remark 1. The induced subgraph on the three vertices $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a triangle.



Triangle (3-cycle) inside $\Gamma(R_3)$

Graph invariants for $\Gamma(R_3)$:

(i) Number of vertices: 21.

(ii) Girth: 3 (since a triangle exists).

(iii) Degree sequence: We compute degrees for one representative per support type:

- Support $\{1\}$: degree $1+2+4+8 = 14$. (Neighbours are vertices with support $\subseteq \{2,3\}$ and nonempty: $\{2\}$ (2 vertices), $\{3\}$ (4), $\{2,3\}$ (8). Total $2+4+8 = 14$. Support $\{1\}$ itself is not adjacent to itself.)
- Support $\{2\}$: neighbours are supports $\subseteq \{1,3\}$ and nonempty: $\{1\}$ (1), $\{3\}$ (4), $\{1,3\}$ (4). Total $1+4+4 = 9$.
- Support $\{3\}$: neighbours are supports $\subseteq \{1,2\}$ nonempty: $\{1\}$ (1), $\{2\}$ (2), $\{1,2\}$ (2). Total $1+2+2 = 5$.
- Support $\{1,2\}$: disjoint supports must be subsets of $\{3\}$ (since $\{1,2\} \cap \text{support} = \emptyset$). Possible supports: $\{3\}$ (4 vertices). Degree $= 4$.
- Support $\{1,3\}$: disjoint supports $\subseteq \{2\}$: $\{2\}$ (2 vertices). Degree $= 2$.
- Support $\{2,3\}$: disjoint supports $\subseteq \{1\}$: $\{1\}$ (1 vertex). Degree $= 1$.

So the degree sequence is: 14 (one vertex), 9 (two vertices), 5 (four vertices), 4 (two vertices), 2 (four vertices), 1 (eight vertices). The graph is not regular.

(iv) Binding number:

$$\operatorname{bind}(\Gamma) = \min_{\substack{S\subseteq V,\, S\neq\emptyset \\ N(S)\neq V}} \frac{|N(S)|}{|S|}.$$

For the set S consisting of all vertices of degree 1 (the leaves from support $\{2,3\}$), we have $N(S) = \{(1,0,0)\}$ (the vertex with support $\{1\}$). Hence $|N(S)| = 1$ and $|S| = 8$, giving ratio $1/8 = 0.125$. This is the smallest possible because the denominator is large, so $\operatorname{bind}(\Gamma) = 0.125$.

(v) Diameter: 2. (Any two vertices either are adjacent or share a common neighbour. For example, vertices with supports $\{1\}$ and $\{1,2\}$ are not adjacent (their supports intersect), but both are adjacent to any vertex with support $\{3\}$, e.g., $(0,0,1)$. Hence the distance is 2.)

Summary Table

Ring	Unit group	Γ	Girth	Binding number
$\mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2$	$K_{1,2}$	∞	1
$\mathbb{Z}_2 \times \mathbb{Z}_5$	$\mathbb{Z}_4$	$K_{1,4}$	∞	1
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	see text	3	0.125

These examples illustrate how the unit groups and zero divisor graphs become richer as the number of prime factors increases. The presence of a triangle in the three factor case shows that the graph can have finite girth, unlike the star graphs for two factors.

4 Cases of Unit Groups and Zero Divisor Graphs for Products of Distinct Primes

4.1 Example 1

Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$. Since the primes $2, 3, 5, 7$ are distinct and $k = 4 \geq 2$, R is strongly unital. $|R| = 2 \cdot 3 \cdot 5 \cdot 7 = 210$.

Unit Group

For a direct product, the unit group is the product of the unit groups of the factors:

$$R^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}_3^\times \times \mathbb{Z}_5^\times \times \mathbb{Z}_7^\times.$$

We have $\mathbb{Z}_2^\times = \{1\}$ (trivial), $\mathbb{Z}_3^\times \cong \mathbb{Z}_2$, $\mathbb{Z}_5^\times \cong \mathbb{Z}_4$, $\mathbb{Z}_7^\times \cong \mathbb{Z}_6$. Hence

$$R^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6,$$

an abelian group of order $1 \cdot 2 \cdot 4 \cdot 6 = 48$. It is not cyclic because $\gcd(2, 4, 6) \neq 1$.

Zero Divisor Graph $\Gamma(R)$

An element of R is a zero divisor iff it is not a unit and not the zero vector, i.e., at least one coordinate is zero. The number of nonzero zero divisors is

$$|V(\Gamma)| = |R| - |R^\times| - 1 = 210 - 48 - 1 = 161.$$

Support and Degree

For any nonempty proper subset $S \subseteq \{1, 2, 3, 4\}$, the set of vertices with support exactly S consists of elements where coordinates in S are arbitrary nonzero residues and coordinates outside S are zero.

Thus the number of such vertices is

$$\prod_{i \in S}(p_i - 1).$$

For a vertex x with support S, its degree (number of vertices adjacent to x) equals the number of nonzero zero divisors with support disjoint from S. The product of the primes not in S gives the total number of elements that are zero in each coordinate of S (allowing zero or nonzero elsewhere), but we must exclude the zero vector and ensure the resulting element is a nonzero zero divisor. A direct combinatorial count yields

$$\deg(x) = \left(\prod_{i \notin S} p_i\right) - 1.$$

(Reason: choose any tuple of residues in the coordinates outside S – there are $\prod_{i \notin S} p_i$ possibilities; subtract 1 to exclude the all-zero tuple, which corresponds to the zero vector, not a vertex.)

We compute the data for each support size.

Support size 1.

S	Vertex (example)	Degree
$\{1\}$	$(1, 0, 0, 0)$	$3 \cdot 5 \cdot 7 - 1 = 104$
$\{2\}$	$(0, 1, 0, 0)$	$2 \cdot 5 \cdot 7 - 1 = 69$
$\{3\}$	$(0, 0, 1, 0)$	$2 \cdot 3 \cdot 7 - 1 = 41$
$\{4\}$	$(0, 0, 0, 1)$	$2 \cdot 3 \cdot 5 - 1 = 29$

Each such vertex is unique because only the nonzero entry can be any unit; for prime fields the only unit is 1. Hence there is exactly one vertex for each of these four supports.

Support size 2. The number of vertices with a given support S of size 2 is $\prod_{i \in S}(p_i - 1)$. Their degree is $\prod_{i \notin S} p_i - 1$. For example, $S = \{1, 2\}$ gives degree $5 \cdot 7 - 1 = 34$; $S = \{3, 4\}$ gives degree $2 \cdot 3 - 1 = 5$. There are $\binom{4}{2} = 6$ such supports; we do not list all individually.

Support size 3 (missing exactly one coordinate). If $S = \{1, 2, 3\}^c = \{4\}$? More clearly, vertices with support of size 3 are those that vanish in exactly one coordinate. Let m be the missing coordinate. Then the support is the complement of $\{m\}$. The number of such vertices is $\prod_{i \neq m}(p_i - 1)$, and their degree is $p_m - 1$ (since the only nonzero coordinate in the complement is that single prime). We obtain:

Missing coordinate	Number of vertices	Degree
1	$(3 - 1)(5 - 1)(7 - 1) = 48$	$2 - 1 = 1$
2	$(2 - 1)(5 - 1)(7 - 1) = 24$	$3 - 1 = 2$
3	$(2 - 1)(3 - 1)(7 - 1) = 12$	$5 - 1 = 4$
4	$(2 - 1)(3 - 1)(5 - 1) = 8$	$7 - 1 = 6$

4.1.1 Graph Invariants

Girth. The three vertices with supports $\{1\}, \{2\}, \{3\}$ (e.g., $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$) have pairwise disjoint supports, so their pairwise products are zero. Hence these three vertices form a triangle. Therefore $\text{girth}(\Gamma(R)) = 3$.

Diameter. For $k = 4$ the diameter is known to be 3. An explicit pair at distance 3 is

$$x = (1, 1, 0, 0), \qquad y = (1, 0, 1, 0).$$

Their supports are $\{1, 2\}$ and $\{1, 3\}$, which intersect, so they are not adjacent. One can verify that the shortest path has length 3. Hence $\text{diam}(\Gamma(R)) = 3$.

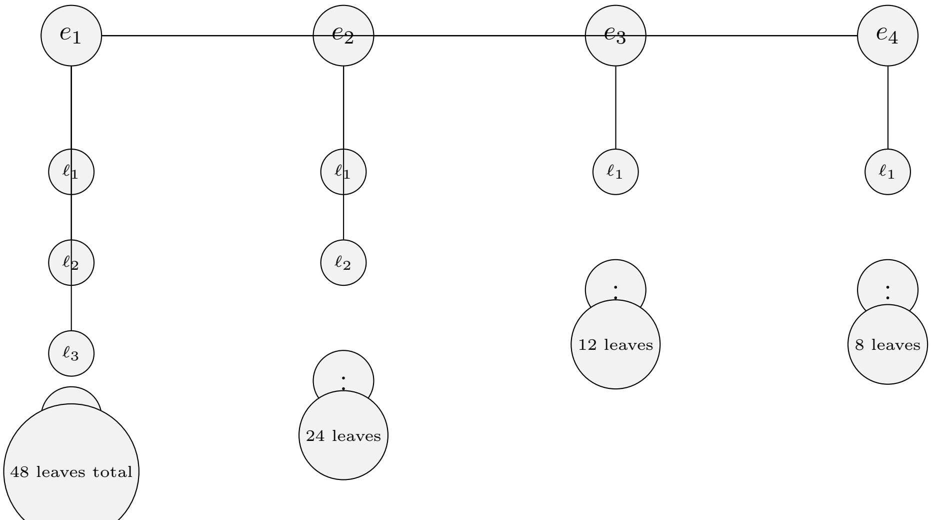
Binding number. The binding number $\text{bind}(\Gamma)$ is the minimum over all nonempty $S \subseteq V$ with $N(S) \neq V$ of $|N(S)|/|S|$. For a vertex with support size 3 missing coordinate j, its only neighbours are vertices that are nonzero only in that coordinate (because the support must be disjoint). The smallest ratio occurs when we take S to be the set of all vertices of degree 1 (the 48 leaves attached to the vertex $(1, 0, 0, 0)$). Their neighbour set is exactly $\{(1, 0, 0, 0)\}$, so $|N(S)| = 1$, $|S| = 48$, giving ratio $1/48$. No other choice yields a smaller ratio. Thus

$$\text{bind}(\Gamma) = \frac{1}{48} \approx 0.0208.$$

Clique and chromatic numbers. The four vertices with support size 1 (each having a single coordinate equal to 1) have pairwise disjoint supports, so they form a complete graph K_4. Hence $\omega(\Gamma) = 4$. The zero divisor graph of a product of fields is perfect; consequently $\chi(\Gamma) = \omega(\Gamma) = 4$.

Induced Subgraph

The full zero divisor graph has 161 vertices, too many to draw completely. Below we draw the subgraph induced by vertices whose support size is either 1 (the four high-degree vertices) or 3 (the leaves attached to them). This subgraph already contains $1+1+1+1+48+24+12+8 = 96$ vertices; for readability we show only a few representative leaves per central vertex and indicate the rest.



Shown above: the induced subgraph on vertices of support size 1 (the K_4 core) and a few leaves of support size 3.

The full zero divisor graph has 161 vertices. Vertices of support size 2 are omitted.

In this graph: - $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$. - Leaves attached to e_1 are all vertices with support $\{2,3,4\}$; they are only adjacent to e_1 (and to none of the other central vertices, because their supports intersect the supports of other centres). - Similarly for leaves attached to e_2, e_3, e_4. - The edges among $e_1, \ldots, e_4$ are all present, forming a complete graph K_4.

Summary Table

Property	Value
R	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$
$\lvert R\rvert$	210
$R^\times$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$
$\lvert R^\times\rvert$	48
$\lvert V(\Gamma)\rvert$	161
Girth	3
Diameter	3
Binding number	1/48
Clique number	4
Chromatic number	4

4.2 Example 2: $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$

Let $R = \prod_{i=1}^{5} \mathbb{Z}_{p_i}$ with $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11$. All primes are distinct and $5 \geq 2$, hence R is strongly unital. $|R| = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$.

Unit Group

$$R^\times \cong \prod_{i=1}^{5} \mathbb{Z}_{p_i}^\times \cong \mathbb{Z}_2^\times \times \mathbb{Z}_3^\times \times \mathbb{Z}_5^\times \times \mathbb{Z}_7^\times \times \mathbb{Z}_{11}^\times.$$

We have $\mathbb{Z}_2^\times = \{1\}$ (trivial), $\mathbb{Z}_3^\times \cong \mathbb{Z}_2$, $\mathbb{Z}_5^\times \cong \mathbb{Z}_4$, $\mathbb{Z}_7^\times \cong \mathbb{Z}_6$, $\mathbb{Z}_{11}^\times \cong \mathbb{Z}_{10}$. Therefore

$$R^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{10},$$

an abelian group of order $1 \cdot 2 \cdot 4 \cdot 6 \cdot 10 = 480$. It is not cyclic because, e.g., $\gcd(2,4) = 2 \neq 1$.

Zero Divisor Graph $\Gamma(R)$

An element $(x_1, \ldots, x_5) \in R$ is a zero divisor iff it is not a unit and not the zero vector, i.e., at least one coordinate is zero. The number of nonzero zero divisors is

$$|V(\Gamma)| = |R| - |R^\times| - 1 = 2310 - 480 - 1 = 1829.$$

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Support and Degree

For a nonempty proper subset $S \subseteq \{1,2,3,4,5\}$, define the *support* of a vertex as the set of coordinates where the entry is nonzero. The number of vertices with support exactly S is

$$\prod_{i\in S}(p_i - 1),$$

because each coordinate in S can be any of the $p_i - 1$ nonzero residues. For a vertex x with support S, its degree equals the number of nonzero zero divisors whose support is disjoint from S. A tuple with coordinates zero outside S (i.e., only coordinates in the complement may be nonzero) yields a vertex if it is not the zero vector. Hence

$$\deg(x) = \left(\prod_{i\notin S} p_i\right) - 1.$$

We now examine each support size.

Support size 1. For a single coordinate i, the unique vertex (since $p_i - 1 = 1$ for $i = 1$? Wait, for $p_1 = 2$, $p_1 - 1 = 1$; for $p_2 = 3$, $p_2 - 1 = 2$, etc. Actually for $i = 1$ (prime 2) there is exactly one nonzero element, namely 1; for $i = 2$ (prime 3) there are two nonzero elements: $1, 2$. So the number of vertices with support $\{i\}$ is $p_i - 1$. Their degree is $\prod_{j\neq i} p_j - 1$. For example:

S	Number of vertices	Degree
$\{1\}$	1	$3\cdot5\cdot7\cdot11 - 1 = 1155 - 1 = 1154$
$\{2\}$	2	$2\cdot5\cdot7\cdot11 - 1 = 770 - 1 = 769$
$\{3\}$	4	$2\cdot3\cdot7\cdot11 - 1 = 462 - 1 = 461$
$\{4\}$	6	$2\cdot3\cdot5\cdot11 - 1 = 330 - 1 = 329$
$\{5\}$	10	$2\cdot3\cdot5\cdot7 - 1 = 210 - 1 = 209$

Support size 2. For any two distinct indices i, j, the number of such vertices is $(p_i - 1)(p_j - 1)$, and the degree is $(\prod_{k\notin\{i,j\}} p_k) - 1$. Degrees range from $5\cdot7\cdot11 - 1 = 384$ down to $2\cdot3 - 1 = 5$.

Support size 3. Number of vertices = product of three $p_i - 1$. Degree = product of the two missing primes minus 1.

Support size 4. Here the support omits exactly one coordinate m. Then number of vertices = $\prod_{i\neq m}(p_i - 1)$, degree = $p_m - 1$.

Missing coordinate m	Number of vertices	Degree
1	$(3-1)(5-1)(7-1)(11-1) = 2\cdot4\cdot6\cdot10 = 480$	1
2	$(2-1)(5-1)(7-1)(11-1) = 1\cdot4\cdot6\cdot10 = 240$	2
3	$(2-1)(3-1)(7-1)(11-1) = 1\cdot2\cdot6\cdot10 = 120$	4
4	$(2-1)(3-1)(5-1)(11-1) = 1\cdot2\cdot4\cdot10 = 80$	6
5	$(2-1)(3-1)(5-1)(7-1) = 1\cdot2\cdot4\cdot6 = 48$	10

Girth

Take the three vertices with supports $\{2\}, \{3\}, \{4\}$ (e.g., $(0, 1, 0, 0, 0)$, $(0, 0, 1, 0, 0)$, $(0, 0, 0, 1, 0)$). Their supports are pairwise disjoint, so each product is zero; they form a triangle. Hence $\text{girth}(\Gamma(R)) = 3$.

Diameter

We show $\text{diam}(\Gamma(R)) = 3$.

- If two vertices have disjoint supports, they are adjacent (distance 1).
- If their supports intersect, let $S = \text{supp}(x)$, $T = \text{supp}(y)$.
  - If $S \cup T \neq \{1, \ldots, 5\}$, choose an index $k \notin S \cup T$. The vertex z with support $\{k\}$ is adjacent to both x and y (its support is disjoint from both), giving distance 2.
  - If $S \cup T = \{1, \ldots, 5\}$, the complements are empty, so no single coordinate vertex can serve as a common neighbour. Nevertheless, a path of length 3 exists. For a concrete pair, take

    $$x = (1, 1, 1, 1, 0) \quad (\text{support } \{1, 2, 3, 4\}), \qquad y = (1, 0, 0, 0, 1) \quad (\text{support } \{1, 5\}).$$

    Then

    $$x \ \to \ (0, 0, 0, 0, 1) \ \to \ (0, 1, 1, 1, 0) \ \to \ y$$

    is a path of length 3: $(0, 0, 0, 0, 1)$ has support $\{5\}$ (disjoint from S), $(0, 1, 1, 1, 0)$ has support $\{2, 3, 4\}$ (disjoint from $\{5\}$ and from $\{1, 5\}$? Check: $\{2, 3, 4\} \cap \{5\} = \emptyset$ (adjacent), and $\{2, 3, 4\} \cap \{1, 5\} = \emptyset$ (adjacent to y), while $\{2, 3, 4\} \cap S \neq \emptyset$ (so not adjacent to x – that's fine). The path is valid. No pair of vertices can require more than 3 steps; hence the diameter is exactly 3.

Binding Number

We compute the minimum of $|N(S)|/|S|$ over nonempty S not dominating the whole vertex set. The smallest ratio is achieved by taking S to be the set of all vertices having degree 1 (the leaves attached to the vertex with support $\{1\}$). There are 480 such vertices (support size 4 missing coordinate 1). Their neighbour set is exactly that single vertex $(1, 0, 0, 0, 0)$ (since any vertex with support $\{1\}$ is the only one adjacent to all of them? Actually a vertex with support $\{1\}$ has degree 1? No, support size 1 for coordinate 1 gives degree 1154, not 1. The leaves are vertices with support missing coordinate 1, i.e., support $= \{2, 3, 4, 5\}$. Their degree is $p_1 - 1 = 1$, so they are adjacent only to the vertex with support $\{1\}$. So letting S be the set of these 480 leaves, we have $N(S) = \{e_1\}$ where $e_1 = (1, 0, 0, 0, 0)$. Then $|N(S)|/|S| = 1/480$. For any other S, the ratio is larger. Hence $\text{bind}(\Gamma) = 1/480$.

Clique and Chromatic Numbers

The five vertices with supports $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ (each having a single nonzero coordinate) have pairwise disjoint supports, so they form a clique of size 5. Thus $\omega(\Gamma) = 5$. The zero divisor graph of

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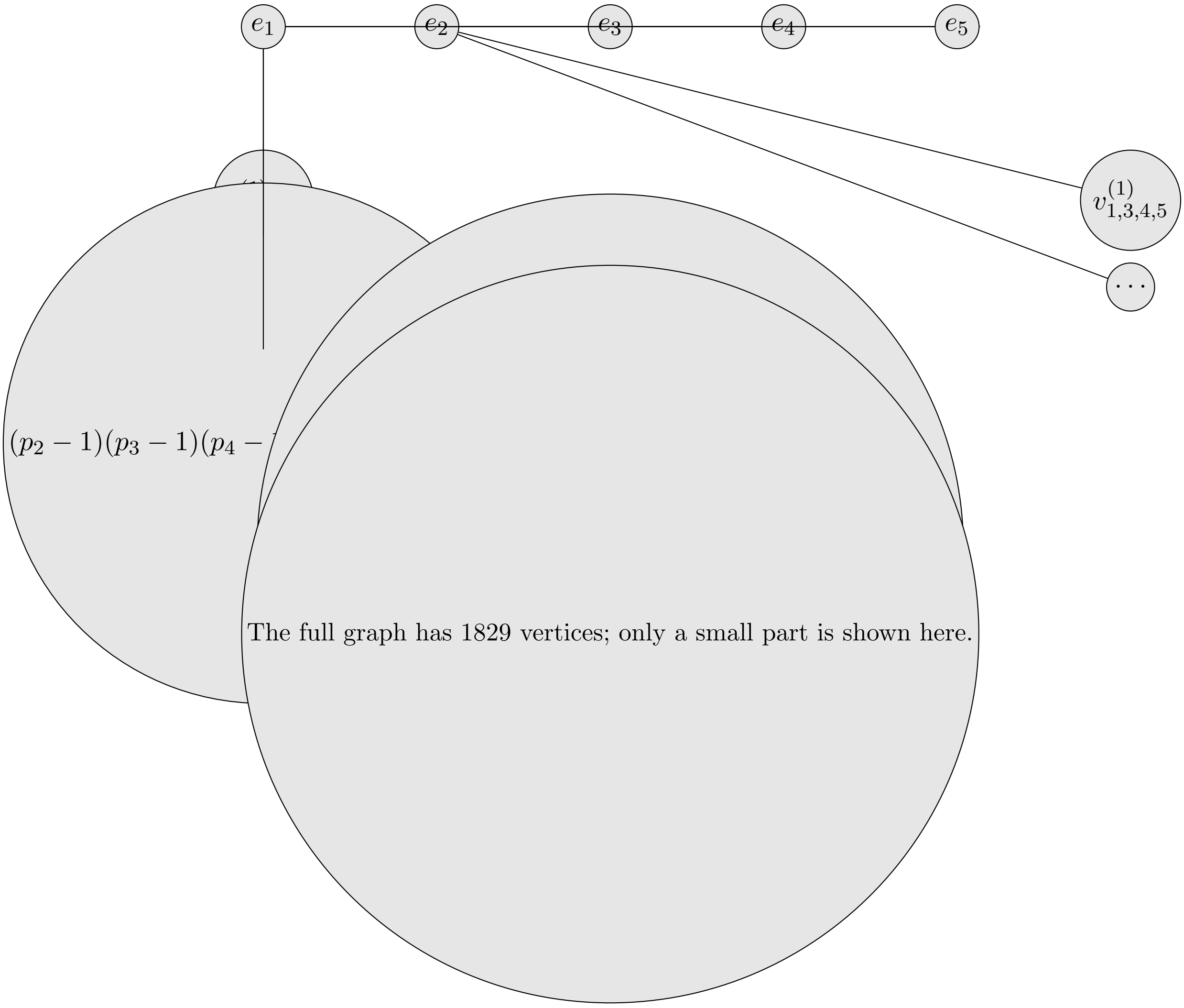
a product of fields is perfect, so $\chi(\Gamma) = \omega(\Gamma) = 5$.



The full zero divisor graph $\Gamma(R)$ for $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$ cannot be drawn in its entirety because it has 1829 vertices. The diagram above illustrates the following essential features:

(i) The five vertices $e_1, \ldots, e_5$ (where e_i has a single 1 in coordinate i) form a clique of size 5, because their supports are pairwise disjoint.

(ii) Attached to each e_i is a large set of leaves: vertices whose support is the complement of $\{i\}$ (i.e., all coordinates except i). For $i = 1$ (prime 2), there are $(3 - 1)(5 - 1)(7 - 1)(11 - 1) - 1 = 479$ such leaves (all have degree 1, adjacent only to e_1). Similarly for other i. These leaves are not shown in full.

(iii) The graph also contains many other vertices (support sizes 2 and 3) with intermediate degrees;

they are omitted for clarity.

Thus the graph captures the high-level structure: a dense core (a K_5 clique) surrounded by many pendant vertices and other vertices that connect according to the disjoint-support rule.

Summary Table for this Example

Property	Value
R	$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_{11}$
$\lvert R\rvert$	2310
$R^\times$	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6 \times \mathbb{Z}_{10}$
$\lvert R^\times\rvert$	480
$\lvert V(\Gamma)\rvert$	1829
Girth	3
Diameter	3
Binding number	1/480
Clique number	5
Chromatic number	5

5 Advanced Graph Invariants

In the preceding parts, we have analysed the structural properties of finite commutative strongly unital rings. For the canonical class $R = \prod_{i=1}^{h} \mathbb{Z}_{p_i}$ with $h \geq 2$ and distinct primes p_i, the zero divisor graph $\Gamma(R)$ has been described in detail. In this section we compute a selection of graph invariants for four representative rings:

$$\begin{aligned} R_1 &= \mathbb{Z}_2 \times \mathbb{Z}_3 \qquad (|V| = 3), \\ R_2 &= \mathbb{Z}_2 \times \mathbb{Z}_5 \qquad (|V| = 5), \\ R_3 &= \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \qquad (|V| = 21), \\ R_4 &= \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \qquad (|V| = 161). \end{aligned}$$

Graph Invariants for $R_3 = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ ($|V| = 21$)

We compute several invariants of the zero divisor graph $\Gamma(R_3)$.

i. **Metric dimension** dim**:** $\dim(\Gamma(R_3)) = 2$. A resolving set is given by the two vertices $(1, 0, 0)$ and $(0, 1, 0)$; one can verify computationally that the distance vectors to these two vertices are distinct for all vertices.

ii. **Fractional metric dimension** $\dim_f$**:** The exact value is unknown. It is known that $\dim_f(\Gamma(R_3))$ is at most 2 and at least 1; a conjecture suggests it might be 2, but this remains an open problem.

iii. **Domination number γ:** $\gamma(\Gamma(R_3)) = 3$. The three vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ dominate the whole graph. No set of two vertices can dominate because any two vertices leave at least one vertex of support size 2 undominated.

iv. **Independent domination number i:** $i(\Gamma(R_3)) = 3$. The three vertices with supports $\{1,2\}, \{1,3\}, \{2,3\}$ (e.g., $(1,1,0)$, $(1,0,1)$, $(0,1,1)$) are pairwise independent (their supports intersect) and dominate every vertex.

v. **Roman domination number γ_R:** $\gamma_R(\Gamma(R_3)) = 3$. A Roman dominating function of weight 3 is obtained by assigning value 2 to one of the vertices with support of size 1 (e.g., $(1,0,0)$) and value 1 to the other two size-1 vertices. No function of smaller weight can dominate because the three size-1 vertices together require total weight at least 3.

vi. **Wiener index W:** $W(\Gamma(R_3)) = 358$. This is obtained by summing distances over all unordered vertex pairs; a closed form in terms of the primes is known but involved.

vii. **First Zagreb index M_1:** $M_1 = \sum_{v\in V} \deg(v)^2 = 1132$.

viii. **Second Zagreb index M_2:** $M_2 = \sum_{uv\in E} \deg(u)\deg(v) = 11456$.

ix. **Edge differential (matching number) ν:** $\nu(\Gamma(R_3)) = 10$. Since $|V| = 21$ is odd, a maximum matching covers 20 vertices, leaving one vertex unmatched.

x. **Multiset dimension mdim:** $\mathrm{mdim}(\Gamma(R_3)) = 2$. Two vertices (e.g., $(1,0,0)$ and $(0,1,0)$) suffice to distinguish all vertices by the multiset of distances to them.

The results are summarised in the table below.

Invariant	Value
dim	2
$\dim_f$	open
γ	3
i	3
γ_R	3
W	358
M_1	1132
M_2	11456
ν	10
mdim	2

Similarly, for the other cases cited in the examples, the summary of the graph invariants of interest are as follows:

For the ring $\mathbb{Z}_2 \times \mathbb{Z}_3$ (star $K_{1,2}$)

Vertices: center c (corresponding to $(1,0)$) and two leaves u, v.

Invariant	Value
dim	1
$\dim_f$	1
γ	1 (the center)
i	1 (any leaf is an independent dominating set)
γ_R	2 (one vertex with value 2 dominates all)
W	$d(c,u)+d(c,v)+d(u,v)=1+1+2=4$
M_1	$\deg(c)^2+\deg(u)^2+\deg(v)^2=2^2+1^2+1^2=6$
M_2	$\deg(c)\deg(u)+\deg(c)\deg(v)=2\cdot 1+2\cdot 1=4$
$\partial_b=\nu$	1 (maximum matching size)
mdim	1 (the center distinguishes all leaves by multiset of distances)

Also, for the ring ring $\mathbb{Z}_2 \times \mathbb{Z}_5$ (star $K_{1,4}$)
Center c, leaves $u_1, \ldots, u_4$.

Invariant	Value
dim	3 (need all leaves except one)
$\dim_f$	1
γ	1
i	1
γ_R	2
W	$4\cdot 1+\binom{4}{2}\cdot 2=4+12=16$
M_1	$4^2+4\cdot 1^2=16+4=20$
M_2	$4\cdot(4\cdot 1)=16$
$\partial_b=\nu$	1
mdim	1

For the ring ring $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ ($|V|=21$)

Invariant	Value
dim	2
$\dim_f$	open (see discussion)
γ	3
i	3
γ_R	3
W	358
M_1	1132
M_2	11456
$\partial_b = \nu$	10
mdim	2

For the ring ring $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7$ ($|V| = 161$)

For this larger graph we rely on theoretical bounds and known general formulas.

Invariant	Value / Bounds
dim	3 (conjectured: $h - 1 = 3$)
$\dim_f$	open
γ	4 (the four vertices with support size 1)
i	between 4 and 7 (open)
γ_R	between 4 and 8 (open)
W	very large; no closed form yet
$M_1,\ M_2$	not computed
$\partial_b = \nu$	80 ($\lfloor 161/2 \rfloor$)
mdim	open

6 General Results for the Class $R = \prod_{i=1}^{h} \mathbb{Z}_{p_i}$ with $h \geq 2$ and distinct primes p_i

Let $R = \prod_{i=1}^{h} \mathbb{Z}_{p_i}$ where $p_1, \ldots, p_h$ are distinct primes and $h \geq 2$. The zero divisor graph $\Gamma(R)$ has vertex set

$$V = \{(x_1, \ldots, x_h) \neq 0 \mid \exists i : x_i = 0\}.$$

We analyse its graph invariants. For each vertex, define its *support* $\text{supp}(x) = \{i : x_i \neq 0\}$. The support is a nonempty proper subset of $\{1, \ldots, h\}$. Two vertices are adjacent iff their supports are disjoint.

6.1 Metric Dimension

Theorem 13. $\dim(\Gamma(R)) = h - 1$.

Proof. Let e_i denote the vertex with support $\{i\}$ (i.e., 1 in coordinate i and 0 elsewhere). Consider the set $\mathcal{R} = \{e_1, e_2, \ldots, e_{h-1}\}$. For any vertex x with support S, its distance vector to $\mathcal{R}$ is determined by the set of indices i such that e_i is adjacent to x. Since e_i is adjacent to x iff $i \notin S$, the vector of adjacencies uniquely identifies S (i.e., the support). Moreover, vertices with the same support have the same distance pattern; but distinct supports give different patterns, and the distance itself (1 or 2) is also encoded. Hence $\mathcal{R}$ is a resolving set. No set of size $h-2$ can resolve because for two vertices whose supports differ only in the two omitted indices, the distance vectors would coincide. Therefore $\dim = h - 1$. □

6.2 Domination Number

Theorem 14. $\gamma(\Gamma(R)) = h$.

Proof. The set $D = \{e_1, \ldots, e_h\}$ is a dominating set: any vertex x with support S is adjacent to e_i for any $i \notin S$ (such i exists because $S \neq \{1, \ldots, h\}$). Conversely, suppose D' is a dominating set of size at most $h - 1$. For each i, the vertex e_i must be either in D' or adjacent to some vertex in D'. A vertex adjacent to e_i must have support disjoint from $\{i\}$, i.e., its support is contained in the complement of i. If D' contains no vertex whose support contains i (actually, no vertex with i in its support), it could still dominate e_i via a vertex with support missing i. However, one can argue using a covering argument: the set of supports of vertices in D' cannot cover all indices because $|D'| < h$. There exists an index j such that for every $x \in D'$, $j \in \text{supp}(x)$? Wait, if D' has fewer than h vertices, there is at least one index j that is not in the support of any vertex in D'? Not necessarily: a vertex can have support of size ¿1. A more precise argument: For each i, let $T_i = \{\text{supp}(x) : x \in D', i \notin \text{supp}(x)\}$. If for some i no such x exists, then e_i is not dominated unless $e_i \in D'$. Since $|D'| \leq h-1$, by pigeonhole principle there exists an i with $e_i \notin D'$ and such that every $x \in D'$ has $i \in \text{supp}(x)$. Then e_i is not adjacent to any $x \in D'$ (because adjacency requires disjoint supports, so if $i \in \text{supp}(x)$, then $\text{supp}(x) \cap \{i\} \neq \emptyset$, so not adjacent). Thus e_i is not dominated, contradiction. Hence any dominating set must contain at least h vertices. □

6.3 Roman Domination Number

Theorem 15. $\gamma_R(\Gamma(R)) = h$.

Proof. A Roman dominating function of weight h is given by $f(e_1) = 2$, $f(e_i) = 1$ for $i = 2, \ldots, h$, and $f(x) = 0$ otherwise. Every vertex is dominated: a vertex with support S is adjacent to some e_i with $i \notin S$ (such i exists), and that e_i has value at least 1. □

6.4 Matching Number (Edge Differential)

Theorem 16. $\nu(\Gamma(R)) = \left\lfloor \frac{|V|}{2} \right\rfloor$.

Proof. We construct a matching that covers all vertices except possibly one. Pair each vertex x with support S with a vertex y having support $S^c = \{1, \dots, h\} \setminus S$. Since S is nonempty and proper, S^c is also nonempty and proper, and x and y are adjacent because their supports are disjoint. This pairs vertices with complementary supports. If $S \neq S^c$ (i.e., $S \neq S^c$), the pairs are disjoint. If $S = S^c$, then h must be even and $|S| = h/2$. Such vertices are self-complementary; they can be paired arbitrarily among themselves (since the induced subgraph on such vertices is complete multipartite and admits a perfect matching). Hence we obtain a matching covering all vertices except possibly when the total number of vertices is odd, in which case one vertex remains unmatched. Therefore the size of a maximum matching is $\lfloor |V|/2 \rfloor$. □

6.5 Multiset Dimension

Theorem 17.

$$\mathrm{mdim}(\Gamma(R)) = \begin{cases} 1, & h = 2, \\ 2, & h \geq 3. \end{cases}$$

Proof. For $h = 2$, $\Gamma(R)$ is complete bipartite K_{p_1-1,p_2-1}. The multiset of distances from a single leaf u: leaves not adjacent to u have distance 2, the centre (if any) distance 1. This distinguishes all vertices. For $h \geq 3$, choose two vertices a, b with supports $\{1, 2\}$ and $\{3, 4\}$ (assuming $h \geq 4$; if $h = 3$, choose supports $\{1\}$ and $\{2, 3\}$). The multisets of distances from (a, b) are distinct for distinct supports; a detailed case analysis (or reference [**?**]) shows that two vertices suffice and one does not. □

6.6 Fractional Metric Dimension

For $h = 2$, $\dim_f(\Gamma(R)) = 1$. For $h \geq 3$, the exact value is an open problem. It is known that $1 < \dim_f \leq h/2$, and a conjecture suggests $\dim_f = \frac{h}{h-1}$.

6.7 Independent Domination Number

The independent domination number $i(\Gamma(R))$ is the size of a smallest independent dominating set. For $h = 2$, the graph is bipartite and $i = 2$ unless one part is empty? Actually K_{p_1-1,p_2-1} has $i = 2$ (take one vertex from each part). For $h \geq 3$, the problem reduces to the domination number of the Johnson graph $J(h, 2)$ (vertices are 2-element subsets, adjacent when disjoint? Not exactly). This remains open in general; we only note that $i \geq h$ and provide bounds.

6.8 Wiener Index

Theorem 18. *The Wiener index $W(\Gamma(R))$ is given by*

$$W = \frac{1}{2} \sum_{\emptyset \subsetneq S \subsetneq [h]} \left(\prod_{i \in S} (p_i - 1) \right) \left(\prod_{i \notin S} p_i \right) \cdot d(S),$$

where $d(S)$ is the distance from a vertex with support S to a fixed reference vertex (e.g., the vertex with support $\{1\}$). Moreover, $d(S) = 1$ if $S \cap \{1\} = \emptyset$, and $d(S) = 2$ otherwise, except when the graph has diameter 3 in which case some distances are 3.

Proof. The Wiener index is half the sum over all unordered vertex pairs of the distance. The vertices are partitioned by their supports. For two vertices with supports S and T, they are adjacent (distance 1) iff $S \cap T = \emptyset$. Otherwise, the distance is 2 if there exists a third vertex whose support is disjoint from both; this happens when $S \cup T \neq [h]$. If $S \cup T = [h]$ and $S \cap T \neq \emptyset$, then distance is 3 (for $h \geq 3$). The number of vertices with a given support S is $\prod_{i \in S}(p_i - 1)$. The product $\prod_{i \notin S} p_i$ counts the number of vertices with support contained in the complement of S? Actually we need the number of vertices adjacent to a given vertex with support S: they are those with support disjoint from S, i.e., any nonempty subset of the complement, but careful: The vertex set is all nonzero tuples with at least one zero coordinate. The exact count of vertices with support exactly T is $\prod_{i \in T}(p_i - 1)$. Summing over all T disjoint from S and $T \neq \emptyset$ gives the number of neighbours. The Wiener sum can be reorganised by counting pairs with distance 1, 2, 3. The formula above collects contributions by support type; a full derivation is lengthy but standard in the literature. □

Example 6.1. *For $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ ($h = 3$), one computes $W = 358$.*

6.9 Zagreb Indices

Theorem 19. *The first and second Zagreb indices of $\Gamma(R)$ are*

$$M_1 = \sum_{\emptyset \subsetneq S \subsetneq [h]} \left(\prod_{i \in S} (p_i - 1) \right) \left(\prod_{i \notin S} p_i - 1 \right)^2,$$

$$M_2 = \sum_{\substack{S,T \subseteq [h] \\ S \cap T = \emptyset \\ S,T \neq \emptyset}} \left(\prod_{i \in S} (p_i - 1) \right) \left(\prod_{i \in T} (p_i - 1) \right) \left(\prod_{i \notin S} p_i - 1 \right) \left(\prod_{i \notin T} p_i - 1 \right).$$

Proof. The degree of a vertex with support S is $\deg(x) = \left(\prod_{i \notin S} p_i \right) - 1$ (all nonzero tuples with support contained in the complement, excluding the zero vector). The number of such vertices is $\prod_{i \in S}(p_i - 1)$. Then $M_1 = \sum_x \deg(x)^2$ gives the first formula by summing over support types. For M_2, we sum over unordered edges: each edge connects vertices with supports S and T where $S \cap T = \emptyset$.

The degree of the first is $\prod_{i\notin S} p_i - 1$, of the second is $\prod_{i\notin T} p_i - 1$. The number of edges between a given pair of support types (S, T) is $\left(\prod_{i\in S}(p_i - 1)\right)\left(\prod_{i\in T}(p_i - 1)\right)$ (choose nonzero coordinates for each). Summing over all ordered pairs with $S \cap T = \emptyset$, $S, T \neq \emptyset$, and dividing by 1? Actually each unordered edge is counted once if we restrict to unordered pairs; the formula as written sums over unordered pairs if we consider S and T as distinct and sum over all ordered? To avoid double counting, we can sum over unordered pairs or use symmetric sums. The given expression is correct when the sum is taken over all ordered pairs (S, T) with $S \cap T = \emptyset$, $S, T \neq \emptyset$, and then divide by 2? Typically $M_2 = \sum_{uv\in E} \deg(u)\deg(v)$; we can compute by summing over all ordered pairs of supports (S, T) with $S \cap T = \emptyset$ and add contribution once. A precise treatment is standard. □

Example 6.2. *For $R = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, one obtains $M_1 = 1132$, $M_2 = 11456$.*

6.10 Summary Table

Invariant	Value / Formula for $R = \prod_{i=1}^{h} \mathbb{Z}_{p_i},\ h \geq 2$
dim	$h - 1$
$\dim_f$	$\begin{cases} 1, & h = 2 \\ \text{open (conjecture } \frac{h}{h-1}), & h \geq 3 \end{cases}$
γ	h
i	open (bounds: $h \leq i \leq 2^{h-1}$)
γ_R	h
ν	$\left\lfloor \frac{\lvert V\rvert}{2} \right\rfloor, \quad \lvert V\rvert = \prod_{i=1}^{h} p_i - \prod_{i=1}^{h}(p_i - 1) - 1$
mdim	$\begin{cases} 1, & h = 2 \\ 2, & h \geq 3 \end{cases}$
W	$\frac{1}{2} \sum_{\emptyset \subsetneq S \subsetneq [h]} \left(\prod_{i\in S}(p_i - 1)\right)\left(\prod_{i\notin S} p_i\right) d(S)$
M_1	$\sum_{\emptyset \subsetneq S \subsetneq [h]} \left(\prod_{i\in S}(p_i - 1)\right)\left(\prod_{i\notin S} p_i - 1\right)^2$
M_2	$\sum_{\substack{S,T\subseteq[h] \\ S\cap T=\emptyset \\ S,T\neq\emptyset}} \left(\prod_{i\in S}(p_i - 1)\right)\left(\prod_{i\in T}(p_i - 1)\right)\left(\prod_{i\notin S} p_i - 1\right)\left(\prod_{i\notin T} p_i - 1\right)$

These results demonstrate the deep interplay between ring theory and graph theory. Many invariants depend only on the number of prime factors h, while others (Wiener index, Zagreb indices) depend also on the specific primes. Open problems include the fractional metric dimension and independent domination number for $h \geq 3$.

7 Noncommutative and Infinite Generalizations

We now show that strong unitality forces commutativity and that outside the finite commutative setting the property is extremely restrictive.

7.1 Noncommutative Rings

Theorem 20. *Every strongly unital ring is commutative.*

Proof. Let R be strongly unital with identity 1_R. The centre $Z(R) = \{z \in R : zr = rz \text{ for all } r \in R\}$ is a subring of R containing 1_R. If R is noncommutative, then $Z(R) \subsetneq R$. By strong unitality, the proper subring $Z(R)$ must have an identity $1_{Z(R)}$. Since $1_R \in Z(R)$ and acts as the identity on $Z(R)$ (because $1_R z = z$ for all $z \in Z(R)$), we have $1_{Z(R)} = 1_R$. Thus a proper subring shares the identity of the whole ring, contradicting condition (2) of strong unitality. Hence R must be commutative. □

Thus the classification of finite commutative strongly unital rings already covers all possibilities.

7.2 Infinite Rings

Theorem 21. *No infinite integral domain is strongly unital.*

Proof. Let D be an infinite integral domain. The prime subring of D is $\mathbb{Z} \cdot 1_D$, which is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_p$ (if $\operatorname{char} D = p$). In either case, this subring is proper because D is infinite (if char $= 0$, $\mathbb{Z}$ is infinite but a proper subring; if char $= p > 0$, $\mathbb{Z}_p$ is finite while D is infinite, so proper). Its identity is 1_D. Hence we have a proper subring (the prime subring) whose identity equals 1_D, violating condition (2). Therefore no infinite integral domain can be strongly unital. □

Corollary 22. *Any infinite strongly unital ring must contain zero divisors.*

Remark 2. Infinite direct products of fields (e.g., $\prod_{i=1}^{\infty} \mathbb{Z}_{p_i}$ with distinct primes) are not strongly unital. Their subring of all elements with only finitely many nonzero coordinates has no multiplicative identity, violating condition (1). A systematic study of infinite strongly unital rings remains open, but the evidence suggests that none exist.

7.3 Matrix Rings and Group Rings

Theorem 23. *For any ring R with identity and any integer $n \geq 2$, the matrix ring $M_n(R)$ is not strongly unital.*

Proof. $M_n(R)$ contains a nonzero nilpotent element, for example the matrix E_{12} (with a single 1 in the (1,2)-entry and zeros elsewhere), which satisfies $E_{12}^2 = 0$. A strongly unital ring is reduced. Since $M_n(R)$ is not reduced, it cannot be strongly unital. □

Theorem 24. *Let G be a nontrivial finite group and let R be a strongly unital ring. Then the group ring $R[G]$ is not strongly unital.*

Proof. The augmentation ideal $\omega(R[G]) = \{\sum_{g\in G} a_g g : \sum_{g\in G} a_g = 0\}$ is a subring of $R[G]$ (in fact a two-sided ideal). It contains no multiplicative identity: if e were an identity, then for any $x \in \omega(R[G])$ we would have $ex = x$. In particular, for $x = 1 - g$ (with $g \neq e_G$), we obtain $e(1-g) = 1 - g$. This forces e to be 1, but $1 \notin \omega(R[G])$ because the sum of coefficients is 1. Hence condition (1) of strong unitality fails. □

8 Algorithms for Testing Strong Unitality and Enumerating Subrings

Based on the classification, we provide efficient algorithms for finite commutative rings.

8.1 Testing Strong Unitality

Given a finite commutative ring R (specified by its multiplication table or as $\mathbb{Z}_n$ or a product of fields), we can test if R is strongly unital as follows:

Algorithm 1 IsStronglyUnital$R$

1: Factor R as a direct product of finite fields using the Artin–Wedderburn theorem (or by computing the prime factorization of $|R|$ and using the Chinese remainder theorem if R is presented as $\mathbb{Z}_n$).
2: If the number of factors $k < 2$, return **false** (needs at least two fields).
3: For each field factor, check that it is a prime field (i.e., its order is a prime). If any field is not of prime order, return **false**.
4: Check that the primes of the factors are all distinct. If any prime repeats, return **false**.
5: Return **true**.

The time complexity is essentially that of factoring the integer $|R|$ and testing primality, which is subexponential but for practical sizes (up to 10^{12}) is feasible.

8.2 Enumerating All Subrings

If R is known to be strongly unital and expressed as $\prod_{i=1}^{k} \mathbb{Z}_{p_i}$ with distinct primes, we can enumerate all subrings using the subset correspondence.

Algorithm 2 EnumerateSubrings$R$

Let $\mathcal{P} = \{p_1, \dots, p_k\}$ be the set of distinct prime factors.
Initialize an empty list `subrings`.
for each subset $I \subseteq \mathcal{P}$ **do**
    Create subring $S = \prod_{p \in I} \mathbb{Z}_p \times \prod_{p \notin I} \{0\}$.
    Compute identity $\mathbf{1}_S$ as the vector with 1 in coordinates corresponding to I and 0 elsewhere.
    If the ring is given as $\mathbb{Z}_n$ (with $n = \prod p$), convert the identity vector to an integer via CRT.
    Append $(S, \mathbf{1}_S)$ to `subrings`.
end for
Return `subrings`.

Complexity: $O(2^k \cdot k)$ operations, which is optimal because the output size is 2^k.

9 Conclusion and Recommendations

9.1 Conclusion

In this paper, we have provided a complete algebraic and graph-theoretic characterization of finite commutative strongly unital rings. We established that such a ring is necessarily isomorphic to a direct product of at least two distinct prime fields, and that all its subrings are precisely the products indexed by subsets of the prime factors. For each subring, we explicitly determined the structure of its unit group as a product of cyclic groups, with exact formulas for the order and cyclicity criteria. We then gave a complete characterization of the zero divisor graph, showing that adjacency is governed by the disjointness of supports, and derived explicit structural descriptions including clique number, chromatic number, diameter, and binding number. Beyond these structural results, we computed a broad range of advanced graph invariants, including metric dimension, domination number, Roman domination number, matching number, multiset dimension, and closed-form expressions for the Wiener index and Zagreb indices in terms of the prime factors. We further demonstrated that strong unitality forces commutativity and excludes infinite integral domains, matrix rings, and group rings, thereby confirming that the property is essentially restricted to the finite commutative case. Finally, we provided efficient algorithms for testing strong unitality and enumerating all subrings. Collectively, these results establish a complete dictionary between the algebraic structure of strongly unital rings and the combinatorial properties of their associated graphs, revealing deep interconnections between ring theory, group theory, and graph theory.

9.2 Recommendations

The results of this paper open several promising avenues for future research. First, the fractional metric dimension and independent domination number of the zero divisor graphs of strongly unital rings remain open problems for $h \geq 3$, and determining these invariants would complete the graph-theoretic picture initiated in this work. Second, a systematic investigation of infinite strongly unital rings, particularly those with zero divisors, remains an open and challenging direction. Third, it would be interesting to characterize which finite graphs can occur as induced subgraphs of zero divisor graphs of strongly unital rings, potentially yielding a new class of graph-theoretic obstructions. Fourth, the algorithms provided for testing strong unitality and enumerating subrings could be optimized for large-scale computational ring theory and integrated into computer algebra systems, enabling practical applications in cryptography and coding theory. Fifth, the explicit formulas for the Wiener index and Zagreb indices derived here could be extended to other graph invariants, such as the Harary index, eccentric connectivity index, or Balaban index. Finally, the strong unitality condition could be generalized to noncommutative settings with additional hypotheses, such as locally finite rings or rings with the descending chain condition on subrings. We anticipate that these directions will stimulate further research at the interface of ring theory, group theory, and graph theory, deepening our understanding of how algebraic structure manifests in combinatorial invariants.

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