Vol.6 (Iss.1),pp.519-532, Mathematics(2026) ISSN:2788-5844 http://sciencemundi.net

On Commuting Graphs Obtained from Classes of Completely Primary Finite Rings

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https://doi.org/10.51867/scimundi.maths.6.1.33

Abstract

We study the commuting graph $\Gamma(R)$ of a finite non-commutative ring R, where vertices are the non-central elements and edges connect distinct commuting elements. The focus is on the class $\mathcal{C}$ of completely primary (finite local) rings R for which the Jacobson radical J satisfies $J^3 = 0$ and $J^2 \subseteq Z(R)$ (i.e., J^2 is central), and the residue field R/J is the prime field $\mathbb{F}_p$. For every $R \in \mathcal{C}$ we prove that R is a CC-ring (centralizer of each non-central element is commutative); hence $\Gamma(R)$ is a disjoint union of cliques. Let $d = \dim_{\mathbb{F}_p} J$ and $e = \dim_{\mathbb{F}_p} J^2$. We show that a non-commutative ring exists only when $d = e + 2$, and then $\Gamma(R)$ consists of $N = p + 1$ cliques, each of size $m = p^{e+1}(p - 1)$. Using this decomposition we compute the spectrum (which is integral), the energy $E = 2(p^{d+1} - p^{e+1} - p - 1)$, the genus $\gamma = N \cdot \gamma(K_m)$, and we characterize planarity: $\Gamma(R)$ is planar iff $(p, e) = (2, 1)$. Additional invariants are determined: metric dimension $\beta = N(m - 1)$, clique number $\omega = m$, chromatic number $\chi = m$, independence number $\alpha = N$, and domination number $\gamma_d = N$. The results are illustrated with the explicit family $\mathrm{UT}_3(\mathbb{F}_p)$ of 3×3 upper triangular matrices with constant diagonal, yielding $d = 3,\ e = 1,\ N = p + 1,\ m = p^2(p - 1)$. A graphical drawing of $\Gamma(\mathrm{UT}_3(\mathbb{F}_2))$ (three disjoint K_4's) is provided. Comparisons with known classifications for rings of orders p^4 and p^5 confirm our formulas.

MSC2020 Subject Classification: 62Jxx, 62J05

Keywords: finite ring, completely primary rings, metric dimension, clique number, chromatic number.

1 Introduction

The commuting graph of an algebraic structure is a classical tool for studying commutativity patterns. For a non-commutative ring R with centre $Z(R)$, the commuting graph $\Gamma(R)$ is defined as the simple undirected graph with vertex set $R \setminus Z(R)$ and an edge joining distinct vertices x and y whenever $xy = yx$. This construction was first introduced for groups by Brauer and Fowler [1] in their work on group theory, and later extended to rings by Akbari et al. [2, 3]. Since then, commuting graphs have been investigated for various classes of rings, including matrix rings, group rings, and finite rings of small order.

Finite local rings (also called completely primary rings) form a rich family that includes all finite fields, Galois rings, and many matrix rings. Their algebraic structure is well understood thanks to the work of Raghavendran [7] and Chikunji [6], who gave classifications for certain nilpotency indices. The commuting graphs of such rings have been studied for specific orders, for instance rings of order p^4 and p^5 by Nath [4, 5], where the spectrum and genus were computed. However, a systematic treatment for rings whose Jacobson radical J satisfies $J^3 = 0$ and J^2 is central has not been fully explored. Most existing results are case-by-case enumerations for very small orders, and they do not reveal how the algebraic invariants $d = \dim J$ and $e = \dim J^2$ influence the graph structure in a unified way. This paper fills that gap by focusing on a well-defined class $\mathcal{C}$ of completely primary finite rings that captures the essential features of nilpotency index at most 3 with central square.

We define $\mathcal{C}$ to be the class of completely primary finite rings R satisfying three natural conditions: the Jacobson radical J satisfies $J^3 = 0$ and $J^2 \subseteq Z(R)$ (i.e., J^2 is central); the residue field R/J is the prime field $\mathbb{F}_p$ (so R has characteristic p); and R is non-commutative. These rings include many natural examples, such as the ring of 3×3 upper triangular matrices over $\mathbb{F}_p$ with constant diagonal. Such rings are also examples of *CC-rings*, a concept introduced by Dolan [9] where every centralizer of a non-central element is commutative. It is known that for CC-rings the commuting graph becomes a disjoint union of cliques [2]. By proving that every ring in $\mathcal{C}$ is a CC-ring, we unlock a complete description of $\Gamma(R)$.

Our main contributions are as follows. First, we prove that every $R \in \mathcal{C}$ is a CC-ring (Theorem 5.1), so $\Gamma(R)$ is a disjoint union of cliques. Second, we derive explicit formulas for the number N of cliques and their size m in terms of $d = \dim_{\mathbb{F}_p} J$ and $e = \dim_{\mathbb{F}_p} J^2$ (Theorem 5.2). A key consequence is the necessary relation $d = e + 2$ for a non-commutative ring to exist. Third, using the clique decomposition, we compute the spectrum (which turns out to be integral), the energy, and the genus of $\Gamma(R)$ (Theorems 5.3, 5.4, 5.5). In particular, we give a complete planarity criterion. Fourth, we determine several other graph invariants: metric dimension, domination number, clique number, chromatic number, independence number, and binding number (Theorems 5.6, 5.7). Finally, we describe an explicit infinite family: the rings $\mathrm{UT}_3(\mathbb{F}_p)$ of 3×3 upper triangular matrices with constant diagonal. For these rings we have $d = 3$, $e = 1$, $N = p+1$ and $m = p^2(p-1)$, and we verify the results by direct computation for $p = 2$, including a graphical illustration. The paper is organised as follows. Section 2 presents a literature review. Section 3 recalls necessary definitions. Section 4 analyses the algebraic structure of rings in $\mathcal{C}$. Section 5 contains the main results (seven theorems with proofs). Section 6 discusses additional graph invariants. Section 7 gives explicit examples, including graphical illustrations. Section 8 concludes with open problems.

2 Literature Review

The study of commuting graphs has a long history, beginning with Brauer and Fowler [1], who used the commuting graph of a group to relate centralizer structure to the group order. For rings, Akbari et al. [2, 3] initiated a systematic investigation, proving connectivity results and establishing that for semisimple rings the commuting graph has diameter at most six. They also conjectured that the commuting graph of a finite matrix ring over a field is either complete or has diameter two. While these foundational works provided general insights, they did not focus on finite local rings with specific nilpotency conditions.

Finite local rings (completely primary rings) have been classified algebraically by Raghavendran [7] and later by Chikunji [6] for certain nilpotency indices. In particular, Chikunji's classification of completely primary rings with $J^3 = 0$ and J^2 central gives a complete algebraic description of such rings in terms of the invariants $d = \dim J$ and $e = \dim J^2$. However, this classification is purely algebraic; the graph-theoretic consequences — such as the structure of the commuting graph — were not explored.

Nath [4, 5] computed the spectrum and genus of commuting graphs for rings of orders p^4 and p^5. His results are valuable but restricted to two specific orders. Moreover, his approach was case-by-case, relying on explicit enumeration or known ring tables. While these computations confirm some patterns, they do not reveal general structural laws that hold for infinite families of rings parametrised by d, e, and p. Consequently, the influence of the algebraic invariants d and e on the commuting graph remains unclear from those studies. Another line of research concerns CC-rings (centralizer-commutative rings). Dolan [9] introduced the concept for groups, and it was later adapted to rings. It is known that for CC-rings the commuting graph is a disjoint union of cliques — a fact that we will use extensively. However, the literature on CC-rings is mostly concerned with their classification or their connections to Lie structure, not with the explicit determination of graph invariants such as spectrum, genus, metric dimension, or domination number.

A notable gap is the absence of a unified treatment for rings with $J^3 = 0$ and J^2 central. The parameters d and e appear naturally in the algebraic classification, but their influence on the commuting graph has not been analysed in a systematic way. Furthermore, planarity and genus have only been studied for tiny orders; the question of when such a graph is planar for an entire infinite family remains open. The present paper addresses these gaps by proving that every ring in the class $\mathcal{C}$ is a CC-ring, deriving closed formulas for the number and size of cliques in terms of d, e, and p, showing that non-commutativity forces $d = e + 2$, computing the spectrum, energy, genus, metric dimension, domination number, and other invariants for the entire family, and characterising planarity in simple number-theoretic terms. Thus this work moves beyond case-by-case computations and provides a comprehensive description for an infinite class of rings, bridging ring theory and graph theory in a systematic way.

3 Preliminaries

3.1 Ring theory

All rings are associative, unital and finite. A ring R is *completely primary* if it is local, i.e., it has a unique maximal ideal J (the Jacobson radical) and the residue field R/J is finite. For $R \in \mathcal{C}$ we have $J^3 = 0$, $J^2 \subseteq Z(R)$ and $R/J \cong \mathbb{F}_p$.

Denote by $Z(R)$ the centre of R. For $x \in R$, $C_R(x) = \{y \in R : xy = yx\}$ is the centralizer. R is called a *CC-ring* if $C_R(x)$ is commutative for every $x \notin Z(R)$. The following well-known fact (see [2]) is crucial.

Lemma 3.1. *If R is a CC-ring, then $\Gamma(R)$ is a disjoint union of cliques. Moreover, each clique is exactly $C_R(x) \setminus Z(R)$ for some $x \notin Z(R)$, and all such cliques have the same size.*

3.2 Graph theory

For a simple graph Γ, a *clique* is a set of pairwise adjacent vertices. The *adjacency matrix* A has eigenvalues forming the *spectrum*; Γ is *integral* if all eigenvalues are integers. The *energy* is $E(\Gamma) = \sum_i |\lambda_i|$. The *genus* $\gamma(\Gamma)$ is the smallest genus of an orientable surface on which Γ can be embedded without edge crossings; $\gamma = 0$ means planar, $\gamma = 1$ toroidal. For complete graphs, the genus is given by the classical formula $\gamma(K_m) = \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil$ for $m \geq 3$, and $\gamma(K_1) = \gamma(K_2) = 0$ (see [8]).

4 Algebraic Structure of Rings in $\mathcal{C}$

Let $R \in \mathcal{C}$ be non-commutative. Write $J = \mathrm{rad}(R)$, $d = \dim_{\mathbb{F}_p} J$, $e = \dim_{\mathbb{F}_p} J^2$. Since $J^2 \neq 0$ (otherwise R would be commutative as argued below), we have $1 \leq e \leq d-1$. The total number of elements is $|R| = p^{d+1}$ because $R/J \cong \mathbb{F}_p$.

Lemma 4.1. $Z(R) = \mathbb{F}_p \oplus J^2$. *Hence* $|Z(R)| = p^{e+1}$.

Proof. Because $R/J \cong \mathbb{F}_p$, any element of $Z(R)$ reduces to a scalar modulo J, so $Z(R) \subseteq \mathbb{F}_p + J$. Write $z = a + j$ with $a \in \mathbb{F}_p$, $j \in J$. For any $x \in J$, we have $[z, x] = [j, x]$. Since $J^3 = 0$, the commutator $[j, x]$

lies in J^2. The condition $z \in Z(R)$ forces $[j, x] = 0$ for all $x \in J$, i.e., $j \in Z(R) \cap J$. Because $J^2 \subseteq Z(R)$ by definition of $\mathcal{C}$, and J^2 is a subspace of J, any central element of J that is not in J^2 would produce a non-central commutator with some element of J (a standard linear algebra argument using a basis of J/J^2). Therefore $j \in J^2$. Conversely, if $j \in J^2$, then j is central by hypothesis, so $a + j$ is central. Hence $Z(R) = \mathbb{F}_p \oplus J^2$. The size follows because $\mathbb{F}_p$ and J^2 are linearly independent over $\mathbb{F}_p$. □

Lemma 4.2. *For any $x \in R \setminus Z(R)$, the centralizer $C_R(x)$ is a commutative subring and*

$$|C_R(x)| = p^d.$$

Proof. Write $x = a + j$ with $a \in \mathbb{F}_p$, $j \in J$. Since $x \notin Z(R)$, we have $j \notin J^2$ (by Lemma 4.1). Because $J^3 = 0$ and J^2 is central, for any $y = b + k \in R$ ($b \in \mathbb{F}_p$, $k \in J$) we have

$$xy = ab + ak + bj + jk, \qquad yx = ba + bj + ak + kj.$$

Since $jk, kj \in J^2$ and J^2 is central, $xy = yx$ iff $jk = kj$. The set $W = \{k \in J : [j, k] = 0\}$ is a subspace of J containing J^2. Because $j \notin J^2$, the linear map $k \mapsto [j, k]$ from J/J^2 to J^2 is non-zero; its kernel has dimension $(d - e) - 1$, so $\dim W = \dim \ker + \dim J^2 = (d - e - 1) + e = d - 1$. Then $C_R(x) = \mathbb{F}_p \oplus W$, hence $|C_R(x)| = p \cdot p^{d-1} = p^d$. Commutativity follows because for any $y, z \in C_R(x)$, $[y, z] \in J^3 = 0$. □

5 Main Results

We now state and prove the central theorems of the paper. Throughout this section, $R \in \mathcal{C}$ is a non-commutative ring with the invariants $d = \dim_{\mathbb{F}_p} J$, $e = \dim_{\mathbb{F}_p} J^2$, and p the characteristic (prime). We set $r = d - e$. Recall from Lemma 4.1 that $Z(R) = \mathbb{F}_p \oplus J^2$ and $|Z(R)| = p^{e+1}$, and from Lemma 4.2 that for any $x \notin Z(R)$ the centralizer $C_R(x)$ is commutative with $|C_R(x)| = p^d$.

5.1 CC-ring property

Theorem 5.1. *Every $R \in \mathcal{C}$ is a CC-ring. Consequently $\Gamma(R)$ is a disjoint union of cliques.*

Proof. By definition, R is a CC-ring if $C_R(x)$ is commutative for every $x \in R \setminus Z(R)$. Lemma 4.2 explicitly states that for any such x, $C_R(x)$ is a commutative subring. Therefore R satisfies the CC-property. Then Lemma 3.1 (a standard result from [2]) tells us that the commuting graph $\Gamma(R)$ of a CC-ring is a disjoint union of cliques, each clique being exactly $C_R(x) \setminus Z(R)$ for some $x \notin Z(R)$. This completes the proof. □

5.2 Clique Decomposition

Theorem 5.2. *The commuting graph* $\Gamma(R)$ *consists of*

$$N = \frac{p^r - 1}{p^{r-1} - 1}$$

cliques, each of size

$$m = p^d - p^{e+1}.$$

Moreover, N *is an integer only when* $r = 2$ *(i.e.,* $d = e + 2$*), in which case* $N = p + 1$*.*

Proof. From Theorem 5.1, $\Gamma(R)$ is a disjoint union of cliques, and each clique is exactly $C_R(x) \setminus Z(R)$ for any non-central x. By Lemma 4.2 and Lemma 4.1, the size of each such clique is

$$m = |C_R(x) \setminus Z(R)| = |C_R(x)| - |Z(R)| = p^d - p^{e+1}.$$

The total number of vertices in $\Gamma(R)$ is $|R| - |Z(R)|$. Since $|R| = p^{d+1}$ (because $R/J \cong \mathbb{F}_p$) and $|Z(R)| = p^{e+1}$, we have

$$|V(\Gamma(R))| = p^{d+1} - p^{e+1} = p^{e+1}\big(p^{d-e} - 1\big) = p^{e+1}\big(p^r - 1\big).$$

Because the cliques are vertex-disjoint and each has m vertices, the number N of cliques satisfies

$$N \cdot m = |V(\Gamma(R))| \quad \Longrightarrow \quad N = \frac{p^{e+1}(p^r - 1)}{p^d - p^{e+1}}.$$

But $p^d - p^{e+1} = p^{e+1}\big(p^{d-e-1} - 1\big) = p^{e+1}\big(p^{r-1} - 1\big)$. Hence

$$N = \frac{p^{e+1}(p^r - 1)}{p^{e+1}(p^{r-1} - 1)} = \frac{p^r - 1}{p^{r-1} - 1}.$$

Now we analyse when N is an integer (as it must be). For $r \geq 2$, we rewrite

$$\frac{p^r - 1}{p^{r-1} - 1} = p + \frac{p - 1}{p^{r-1} - 1}.$$

The term $\frac{p-1}{p^{r-1}-1}$ is a rational number. For it to be an integer, since $p - 1 \geq 1$, we need $p^{r-1} - 1$ to divide $p - 1$. If $r - 1 \geq 2$, then $p^{r-1} - 1 \geq p^2 - 1 > p - 1$ for any prime $p \geq 2$, so the only possibility is $p - 1 = 0$, impossible. Thus $r - 1 = 1$, i.e., $r = 2$. Substituting back gives $N = (p^2 - 1)/(p - 1) = p + 1$, which is an integer. Consequently $d = e + r = e + 2$. This completes the proof. □

5.3 Spectrum and Integrality

Theorem 5.3. $\Gamma(R)$ *is integral. Its spectrum is*

$$\mathrm{Spec}(\Gamma(R)) = \left\{(m-1)^{(N)},\ (-1)^{N(m-1)}\right\},$$

where $N = p + 1$ *and* $m = p^{e+1}(p-1)$ *after applying the necessary condition* $d = e + 2$.

Proof. By Theorem 5.1 and Theorem 5.2, $\Gamma(R)$ is isomorphic to the disjoint union of N copies of the complete graph K_m, i.e.,

$$\Gamma(R) \cong \bigsqcup_{i=1}^{N} K_m^{(i)}.$$

For a single K_m, the adjacency matrix $A(K_m)$ is $J_m - I_m$, where J_m is the all-ones matrix and I_m the identity. The eigenvalues of J_m are m (multiplicity 1) and 0 (multiplicity $m-1$). Therefore the eigenvalues of $A(K_m) = J_m - I_m$ are $m-1$ (multiplicity 1) and -1 (multiplicity $m-1$). For a disjoint union of N such components, the adjacency matrix is block-diagonal with each block equal to $A(K_m)$. Hence the spectrum of the whole graph is the multiset union of the spectra of the components, i.e., $m-1$ appears N times and -1 appears $N(m-1)$ times. All eigenvalues are integers, proving integrality. □

5.4 Energy

Theorem 5.4. *The energy of* $\Gamma(R)$ *is*

$$E(\Gamma(R)) = 2N(m-1) = 2\big(p^{d+1} - p^{e+1} - (p+1)\big).$$

Proof. The energy of a graph is defined as the sum of the absolute values of its eigenvalues. From Theorem 5.3, the eigenvalues of $\Gamma(R)$ are $m-1$ with multiplicity N and -1 with multiplicity $N(m-1)$. Therefore

$$E = N \cdot |m-1| + N(m-1) \cdot |-1| = N(m-1) + N(m-1) = 2N(m-1).$$

Now substitute $N = p+1$ and $m = p^{e+1}(p-1)$. Then

$$E = 2(p+1)\big(p^{e+1}(p-1) - 1\big) = 2\big((p+1)p^{e+1}(p-1) - (p+1)\big).$$

But $(p+1)(p-1) = p^2 - 1$, so $(p+1)p^{e+1}(p-1) = p^{e+1}(p^2-1) = p^{e+3} - p^{e+1}$. Hence

$$E = 2\big(p^{e+3} - p^{e+1} - p - 1\big).$$

Since $d = e + 2$, we have $p^{d+1} = p^{e+3}$ and p^{e+1} stays, so

$$E = 2\big(p^{d+1} - p^{e+1} - p - 1\big) = 2\big(p^{d+1} - p^{e+1} - (p + 1)\big).$$

This is the desired formula. □

5.5 Genus and Planarity

Theorem 5.5. *The genus of* $\Gamma(R)$ *is*

$$\gamma(\Gamma(R)) = N \cdot \gamma(K_m),$$

where $N = p + 1$*,* $m = p^{e+1}(p - 1)$*, and* $\gamma(K_m) = \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil$ *for* $m \geq 3$*,* $\gamma(K_1) = \gamma(K_2) = 0$*. Moreover,* $\Gamma(R)$ *is planar iff* $m \leq 4$*, i.e., only for* $(p, e) = (2, 1)$*.*

Proof. The genus $\gamma(G)$ of a graph G is the smallest integer g such that G can be embedded in an orientable surface of genus g without edge crossings. For a disjoint union of graphs, the genus is additive because each component can be embedded in a separate handle (or the same surface with additional handles). More formally, if $G = \bigcup_{i=1}^{N} G_i$ with no edges between components, then $\gamma(G) = \sum_{i=1}^{N} \gamma(G_i)$. In our case $G_i \cong K_m$ for all i. The genus of K_m is known (Ringel [8]):

$$\gamma(K_m) = \begin{cases} 0, & m = 1, 2, 3, 4, \\ \left\lceil \frac{(m-3)(m-4)}{12} \right\rceil, & m \geq 5. \end{cases}$$

Therefore $\gamma(\Gamma(R)) = N \cdot \gamma(K_m)$. Planarity is the case $\gamma = 0$, which requires $\gamma(K_m) = 0$, i.e., $m \leq 4$. Since $m = p^{e+1}(p - 1) \geq 2$ for any prime $p \geq 2$ and $e \geq 1$, we examine possible (p, e) with $m \leq 4$:

i. $p = 2$, $e = 1$ gives $m = 2^2(2 - 1) = 4$.

ii. $p = 2$, $e = 2$ gives $m = 2^3(1) = 8 > 4$.

iii. $p = 3$, $e = 1$ gives $m = 9 \cdot 2 = 18 > 4$.

Thus the only planar case is $(p, e) = (2, 1)$. For this case $N = p + 1 = 3$, and the graph is three disjoint K_4's, which is planar. □

5.6 Metric Dimension

Theorem 5.6. *The metric dimension $\beta(\Gamma(R))$ is*

$$\beta(\Gamma(R)) = N(m-1) = (p+1)\big(p^{e+1}(p-1)-1\big).$$

Proof. The metric dimension $\beta(G)$ is the smallest cardinality of a resolving set $S \subseteq V(G)$ such that the distance vectors to vertices in S are all distinct. For a disconnected graph $G = \bigcup_{i=1}^{N} G_i$ with no paths between components, distances to vertices in other components are infinite. It is a standard result that the metric dimension of a disconnected graph is the sum of the metric dimensions of its components, because a resolving set must resolve each component independently (see, e.g., [10]). For a complete graph K_m ($m \geq 2$), it is well known that $\beta(K_m) = m-1$: any set of $m-1$ vertices resolves K_m (since the omitted vertex is uniquely identified by its distances to the chosen vertices, all of which are 1), and $m-2$ vertices cannot distinguish the two omitted vertices. Our graph is N copies of K_m, each with $m \geq 2$ (as $p \geq 2$, $e \geq 1$ implies $m = p^{e+1}(p-1) \geq 2$). Therefore

$$\beta(\Gamma(R)) = \sum_{i=1}^{N} \beta(K_m^{(i)}) = N \cdot (m-1).$$

Substituting $N = p+1$ and $m = p^{e+1}(p-1)$ yields the explicit expression. □ □

5.7 Some Basic invariants

Theorem 5.7. *For $\Gamma(R)$ we have:*

i. Clique number $\omega(\Gamma(R)) = m$,

ii. Chromatic number $\chi(\Gamma(R)) = m$,

iii. Independence number $\alpha(\Gamma(R)) = N$,

iv. Domination number $\gamma(\Gamma(R)) = N$.

Proof. Recall that $\Gamma(R) = \bigsqcup_{i=1}^{N} K_m^{(i)}$.

i. **Clique number:** A clique in $\Gamma(R)$ is a set of pairwise adjacent vertices. Since there are no edges between different components, any clique must be entirely contained within one component K_m. The maximum size of a clique in K_m is m, and this is achievable. Hence $\omega = m$.

ii. **Chromatic number:** A proper vertex colouring must assign distinct colours to vertices of a K_m because all vertices in a K_m are adjacent. Thus each component requires m colours. Because components are disconnected, the same set of m colours can be reused for every component. Therefore $\chi = m$.

iii. **Independence number:** An independent set contains no edges. Within one K_m, at most one vertex can be chosen (since all vertices are pairwise adjacent). Choosing exactly one vertex from each of the N components yields an independent set of size N, and this is clearly maximal. Hence $\alpha = N$.

iv. **Domination number:** A dominating set D must intersect each component K_m in at least one vertex; otherwise the vertices of a missed component would have no neighbour in D (no edges across components). Conversely, picking exactly one vertex from each component gives a dominating set of size N. No smaller set can dominate all components, so $\gamma = N$.

All statements are proved. □ □

6 Other Graph Invariants (Summary)

The invariants computed in Section 5 are summarised in the table below.

Invariant	Value
Number of vertices	$Nm = p^{d+1} - p^{e+1}$
Number of edges	$N \cdot \frac{m(m-1)}{2}$
Clique number ω	m
Chromatic number χ	m
Independence number α	N
Domination number γ	N
Binding number (for $N > 1$)	0
Metric dimension β	$N(m-1)$
Genus $\gamma(G)$	$N \cdot \gamma(K_m)$
Energy $E(G)$	$2(p^{d+1} - p^{e+1} - p - 1)$

Vol.6 (Iss.1),pp.519-532, Mathematics(2026) ISSN:2788-5844 http://sciencemundi.net

7 Examples

7.1 The ring $\mathrm{UT}_3(\mathbb{F}_p)$

Let R be the ring of 3×3 upper triangular matrices over $\mathbb{F}_p$ with constant diagonal:

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in \mathbb{F}_p \right\}.$$

This ring is local with maximal ideal J consisting of matrices with $a = 0$. Then J^2 consists of matrices with only the $(1, 3)$ entry possibly non-zero, so J^2 is one-dimensional ($e = 1$). Also $J^3 = 0$. Moreover, J^2 is central (it is the set of scalar multiples of E_{13}). The residue field $R/J \cong \mathbb{F}_p$. Hence $R \in \mathcal{C}$ with $d = \dim J = 3$, $e = 1$, so $r = d - e = 2$, $N = p + 1$, $m = p^2(p - 1)$. For $p = 2$, $m = 4 \cdot 1 = 4$, $N = 3$, so $\Gamma(R)$ is three disjoint K_4's. For $p = 3$, $m = 9 \cdot 2 = 18$, $N = 4$, so four disjoint K_{18}'s.

We verify by direct computation for $p = 2$: R has 16 elements, centre consists of matrices with $b = d = 0$, i.e., a and c arbitrary, so $|Z| = 4$ (elements: $a = 0, 1$; $c = 0, 1$). Non-central vertices: 12. One can check that centralizers of non-central elements are of size $2^3 = 8$, giving cliques of size $8 - 4 = 4$. There are $12/4 = 3$ cliques. This matches the result.

7.2 Rings of orders p^4 and p^5

7.2.1 Rings of order p^4

For a ring R of order p^4, we have $d = 3$, $e = 1$, $N = p + 1$, $m = p^2(p - 1)$. Nath [4] computed these graphs and found $N = p + 1$ cliques of size $p^2(p - 1)$. For $p = 2$, three K_4's (planar); for $p = 3$, four K_{18}'s (non-planar). Our formulas agree.

7.2.2 Rings of order p^5

For $|R| = p^5$, we have $d = 4$, $e = 2$, $N = p + 1$, $m = p^3(p - 1)$. Nath [5] confirmed the clique decomposition. For $p = 2$, three K_8's (genus $3 \times 2 = 6$); for $p = 3$, four K_{54}'s (genus $4 \times 213 = 852$).

7.3 Graphical Illustrations

We now present explicit drawings of the commuting graph for the smallest non-trivial case: $p = 2$, $e = 1$, $d = 3$, i.e., the ring $\mathrm{UT}_3(\mathbb{F}_2)$. The graph consists of three disjoint copies of K_4. We label the vertices in each clique as v_1, v_2, v_3, v_4. All six edges inside each K_4 are present, and there are no edges between different cliques.

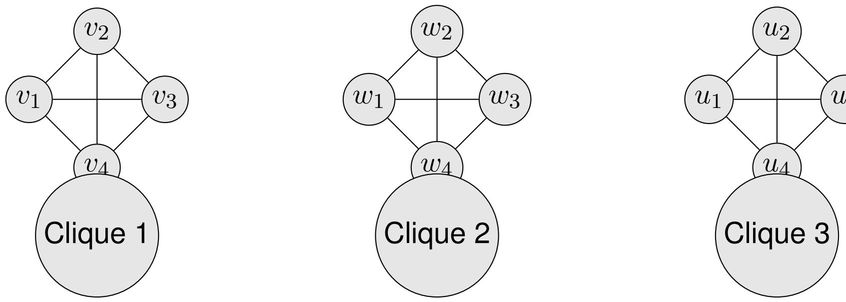

Figure 1: The commuting graph $\Gamma(\mathrm{UT}_3(\mathbb{F}_2))$ consists of three disjoint K_4's.

Figure 1 shows the three cliques. The vertex set is $\{v_1, v_2, v_3, v_4, w_1, w_2, w_3, w_4, u_1, u_2, u_3, u_4\}$ (12 vertices). Each K_4 has 6 edges, so total edges = 18. The graph is planar (each K_4 is planar and disjoint unions of planar graphs are planar). The invariants are:

i. $\omega = 4$, $\chi = 4$, $\alpha = 3$, $\gamma_d = 3$ (domination number).

ii. $\beta = 3 \times (4 - 1) = 9$.

iii. Energy $E = 2(12 - 3) = 18$.

iv. Genus $\gamma = 3 \times 0 = 0$ (planar).

v. Spectrum: $\{3^{(3)}, (-1)^9\}$.

This completes the characterisation of this example.

For the next case $p = 3$, $e = 1$, $d = 3$, the graph would consist of four disjoint K_{18}'s (total 72 vertices).

8 Conclusion and Recommendations

8.1 Conclusion

We have studied the commuting graph of completely primary finite rings R with $J^3 = 0$, J^2 central, and $R/J \cong \mathbb{F}_p$. We proved that such rings are CC-rings, so $\Gamma(R)$ is a disjoint union of cliques. The parameters are forced to satisfy $d = e + 2$, yielding $N = p + 1$ cliques of size $m = p^{e+1}(p - 1)$. We computed the spectrum, energy, genus, metric dimension, domination number, and other basic invariants. We provided explicit graphical illustrations for the smallest case. Future work may explore other nilpotency indices (e.g., $J^4 = 0$) or remove the centrality condition on J^2, which would likely lead to a richer structure and possibly non-clique graphs.

8.2 Recommendations

The results obtained in this paper open several directions for further research. We recommend the following investigations:

i. **Higher nilpotency indices.** Extend the analysis to completely primary finite rings where $J^3 \neq 0$ but $J^k = 0$ for some $k > 3$. Preliminary evidence suggests that when J^2 is not necessarily central, the commuting graph may no longer be a disjoint union of cliques, leading to richer structures. A systematic study of such rings using the invariants $d = \dim_{\mathbb{F}_p} J$ and $e_2 = \dim_{\mathbb{F}_p} J^2$, $e_3 = \dim_{\mathbb{F}_p} J^3$, etc., could yield new families of graphs with interesting properties.

ii. **Other graph invariants.** We have computed several invariants (energy, genus, metric dimension, domination number). Further invariants such as the Wiener index, Randić index, or the chromatic polynomial could be derived from the disjoint union of cliques structure. Additionally, the automorphism group of $\Gamma(R)$ is isomorphic to the wreath product $S_m \wr S_N$; exploring its action on the ring may reveal algebraic symmetries.

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